

Solutions to Homework 4

6.262 Discrete Stochastic Processes

MIT, Spring 2011

Solution to Exercise 2.28:

The purpose of this problem is to illustrate that for an arrival process with independent but not identically distributed interarrival intervals, X_1, X_2, \dots , the number of arrivals $N(t)$ in the interval $(0, t]$ can be a defective rv. In other words, the ‘counting process’ is not a stochastic process according to our definitions. This illustrates that it is necessary to prove that the counting rv’s for a renewal process are actually rv’s.

a) Let the distribution function of the i th interarrival interval for an arrival process be $F_{X_i}(x_i) = 1 - \exp(-\alpha^{-i}x_i)$ for some fixed $\alpha \in (0, 1)$. Let $S_n = X_1 + \dots + X_n$ and show that

$$\mathbb{E}[S_n] = \frac{\alpha(1 - \alpha^n)}{1 - \alpha}.$$

b) Sketch a ‘reasonable’ sample function for $N(t)$.

c) Find $\sigma_{S_n}^2$.

d) Use the Chebyshev inequality on $\Pr S_n \geq t$ to find an upper bound on $\Pr N(t) \leq n$ that is smaller than 1 for all n and for large enough t . Use this to show that $N(t)$ is defective for large enough t .

SOLUTION: **a)** Each X_i is an exponential rv, of rate α^i , so $\mathbb{E}[X_i] = \alpha^i$. Thus

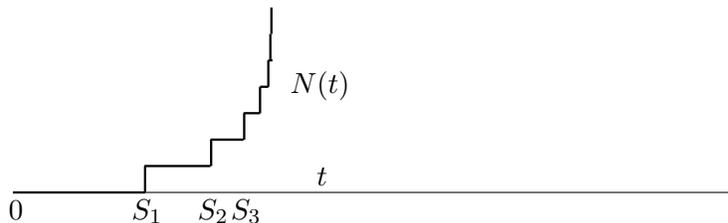
$$\mathbb{E}[S_n] = \alpha + \alpha^2 + \dots + \alpha^n.$$

Recalling that $1 + \alpha + \alpha^2 + \dots = 1/(1 - \alpha)$,

$$\begin{aligned} \mathbb{E}[S_n] &= \alpha(1 + \alpha + \dots + \alpha^{n-1}) \\ &= \frac{\alpha(1 - \alpha^n)}{1 - \alpha} < \frac{\alpha}{1 - \alpha} \end{aligned}$$

In other words, not only is $\mathbb{E}[X_i]$ decaying to 0 geometrically with increasing i , but $\mathbb{E}[S_n]$ is upper bounded, for all n , by $\alpha/(1 - \alpha)$.

b) Since the expected interarrival times are decaying geometrically and the expected arrival epochs are bounded for all n , it is reasonable for a sample path to have the following shape:



Note that the question here is not precise (there are obviously many sample paths, and which are ‘reasonable’ is a matter of interpretation). The reason for drawing such sketches is to acquire understanding to guide the solution to the following parts of the problem.

c) Since X_i is exponential, $\sigma_{X_i}^2 = \alpha^{2i}$. Since the X_i are independent,

$$\begin{aligned}\sigma_{S_n}^2 &= \sigma_{X_1}^2 + \sigma_{X_2}^2 \cdots + \sigma_{X_n}^2 \\ &= \alpha^2 + \alpha^4 + \cdots + \alpha^{2n} \\ &= \alpha^2(1 + \alpha^2 + \cdots + \alpha^{2(n-1)}) \\ &= \frac{\alpha^2(1 - \alpha^{2n})}{1 - \alpha^2} < \frac{\alpha^2}{1 - \alpha^2}\end{aligned}$$

d) The figure suggests (but does not prove) that for typical sample functions (and in particular for a set of sample functions of nonzero probability), $N(t)$ goes to infinity for finite values of t . If the probability that $N(t) \leq n$ (for a given t) is bounded, independent of n , by a number strictly less than 1, then that $N(t)$ is a defective rv rather than a true rv.

The problem suggests using the Chebyshev bound to prove this, but the Markov bound is both simpler and also shows that $N(t)$ is defective for smaller values of t than established by Chebyshev. Thus we first use Markov and then use Chebyshev. By Markov,

$$\begin{aligned}\Pr\{S_n \geq t\} &\leq \frac{\bar{S}_n}{t} \leq \frac{\alpha}{t(1 - \alpha)} \\ \Pr\{N(t) < n\} &= \Pr\{S_n > t\} \leq \Pr\{S_n \geq t\} \leq \frac{\alpha}{t(1 - \alpha)}\end{aligned}$$

where we have used Eq. 2.3 in the text. Since this is valid for all n , it is also valid for $\Pr N(t) \leq n$. For any $t > \alpha/(1 - \alpha)$, we see that $\frac{\alpha}{t(1 - \alpha)} < 1$. Thus $N(t)$ is defective for any such t , i.e., for any t greater than $\lim_{n \rightarrow \infty} \mathbb{E}S_n$.

Now to use Chebyshev, note that

$$\begin{aligned}\Pr\{S_n \geq t\} &= \Pr\{(S_n - \bar{S}_n) \geq t - \bar{S}_n\} \\ &\leq \Pr\{|S_n - \bar{S}_n| \geq t - \bar{S}_n\} \\ &\leq \frac{\sigma_{S_n}^2}{(t - \bar{S}_n)^2}\end{aligned}$$

Now $\sigma_{S_n}^2$ is increasing in n to the limit $\sigma_\infty^2 = \alpha^2/(1 - \alpha^2)$. Also \bar{S}_n is increasing to the limit $\bar{S}_\infty = \alpha/(1 - \alpha)$. Thus $(t - \bar{S}_n)^{-2}$ is also increasing in n , and

$$\Pr\{N(t) > n\} \leq \Pr\{S_n \leq t\} \leq \frac{\sigma_\infty^2}{(t - \bar{S}_\infty)^2} \quad \text{for all } n$$

Thus $\Pr\{N(t) \geq n\}$ is bounded for all n by a number less than 1 (and thus $N(t)$ is defective) if $t - \bar{S}_\infty > \sigma_\infty$, i.e., if $t > \frac{\alpha}{1 - \alpha} + \sigma_\infty$.

Note that this result is weaker (and considerably more complicated) than that using the Markov bound. Actually, by working a good deal harder, it can be shown that $N(t)$ is defective for all $t > 0$. The outline of the argument is as follows: for any given t ,

we choose an m such that $\Pr S_m \leq t/2 > 0$ and such that $\Pr S_\infty - S_m \leq t/2 > 0$ where $S_\infty - S_m = \sum_{i=m+1}^\infty X_i$. The second inequality can be satisfied for m large enough by the Markov inequality. The first inequality is then satisfied since S_m has a density that is positive for $t > 0$.

Solution to Exercise 3.2:

a) Based on the definition of the *recurrent states*, if a state is accessible from all the states that are accessible from it, (i.e., if for all i_2 that $i_1 \rightarrow i_2$, we also have $i_2 \rightarrow i_1$), the state i_1 is called recurrent. So if a state is not recurrent (i.e., it is transient), there should exist at least one state i_2 that is accessible from i_1 ($i_1 \rightarrow i_2$) but i_1 is not accessible from it ($i_2 \not\rightarrow i_1$).

b) The same reasoning as in part **(a)** shows that if i_2 is transient, there should exist a state i_3 where $i_2 \rightarrow i_3$ but $i_3 \not\rightarrow i_2$. Additionally, we know that $i_1 \rightarrow i_2$ and $i_2 \rightarrow i_3$ imply that $i_1 \rightarrow i_3$.

We know that $i_1 \rightarrow i_2$. Now if $i_3 \rightarrow i_1$, then $i_3 \rightarrow i_2$ which is against what we said about i_3 . Then i_1 should not be accessible from i_3 . (i.e., $i_3 \not\rightarrow i_1$)

c) The same reasoning as in part **(a)** shows that if i_k is transient, there should exist a state i_{k+1} for which $i_k \rightarrow i_{k+1}$ but $i_{k+1} \not\rightarrow i_k$. Now if for any $1 \leq j \leq k$, $i_{k+1} \rightarrow j$, then we also have $i_{k+1} \rightarrow i_k$ which is against the assumption about i_k .

d) Each state i_k constructed in part c) is distinct from states i_1, \dots, i_{k-1} , since there is a path from i_j to i_k but not from i_k to i_j . Since there are only J states, the process in part c) must terminate with some $k \leq J$, with means that i_k isn't transient, i.e., is recurrent.

Solution to Exercise 3.3:

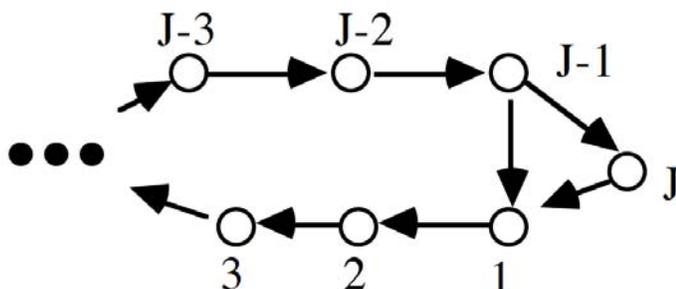
In exercise 3.2 it is proved that each Markov chain has at least one recurrent state which induces a recurrent class of states. State 1 is accessible from all the states. So it is also accessible from the states of the recurrent class in this Markov chain. According to the definition of recurrent class, all the states that are accessible from the states of a recurrent class should have access to the states of this class, meaning that state 1 should communicate to all of the classes of recurrent states. If two recurrent states communicate with one state, they should also communicate with each other, i.e., they belong to a single recurrent class. Hence, there should exist at most one recurrent class of states in this Markov chain.

Problem 4:

The desired graph is shown below. It should be clear that this construction can be used for any number of states $M \geq 3$. Starting in state 1, there is only one possible state (first 2, then 3, up to J) that the chain can be for the first $J - 1$ steps. On the J th step, there are two possible states (1 and 2). For each step from M to $2M - 2$, there are two

possible states (first 1,2, then 2,3, up to $M - 1, M$. Similarly, from steps $2(M - 1) + 1$ to $3(M - 1)$, there are 3 possible states (first 1, 2, 3, then 2, 3, 4, etc.). This continues up to $(M - 1)^2$, which is the last step at which there are $M - 1$ possible states, and then from $(M - 1)^2 + 1$ on, all states are possible at each transition. This pattern should be clear from your tabulation for $M = 4$.

The desired graph is shown below. It should be clear that this construction exists for all $J \geq 3$. Any walk from node J back to node J must be of length $iJ + k(J - 1)$ for some $i \geq 1$ and some $k \geq 1$. Note that $iJ + k(J - 1) = i + (i + k)(J - 1)$. Trying to set this equal to $(J - 1)^2$ yields $i + (i + k - J + 1)(J - 1) = 0$. This implies that i is a multiple $J - 1$, which makes $iJ > (J - 1)^2$, so there is no walk from node J back to itself in $(J - 1)^2$ steps.



Solution to Exercise 3.8:

$[P]$ is doubly stochastic if both $[P]$ and $[P]^T$ are stochastic. This is just a different way of saying that all the rows and all the columns of $[P]$ must sum to one (and all entries are non-negative). Since $[P]$ is stochastic, it has \mathbf{e} (a column vector) as a right eigenvector with the associated eigenvalue $\lambda = 1$, i.e., $[P]\mathbf{e} = \mathbf{e}$. Since $[P]^T$ is stochastic also, $[P]^T\mathbf{e} = \mathbf{e}$. Taking transposes of both sides we get $\mathbf{e}^T[P] = \mathbf{e}^T$, so \mathbf{e}^T is a left eigenvector of $[P]$ associated with $\lambda = 1$. Theorem 3.3.1 guarantees that this left eigenvector is unique within a scale factor, so scaling \mathbf{e}^T to become a probability vector, $\pi = [1/M, 1/M, \dots, 1/M]$.

Solution to Exercise 3.9:

a) The steady state distribution of the Markov chain. is a probability vector π for which $\sum_{j=1}^K \pi_j = 1$ and for each $1 \leq j \leq K$, $\pi_j = \sum_{i=1}^K P_{ij}\pi_i$.

In this Markov chain, We can have the following equations:

$$\begin{aligned} \pi_0 &= (1 - p)\pi_0 + (1 - p)\pi_1 \\ \pi_j &= p\pi_{j-1} + (1 - p)\pi_{j+1}, \text{ for } 1 \leq j \leq k - 2 \\ \pi_{k-1} &= p\pi_{k-2} + p\pi_{k-1} \end{aligned}$$

From the first equations we can have:

$$\pi_1 = \frac{p}{1 - p}\pi_0$$

Using the second equation and $\pi_1 = \frac{p}{1-p}\pi_0$, we can get:

$$\pi_2 = \frac{p^2}{(1-p)^2}\pi_0$$

Similarly, we can get:

$$\pi_j = \frac{p^j}{(1-p)^j}\pi_0$$

for $0 \leq j \leq k-2$.

The last equation says $\pi_{k-1} = \frac{p}{1-p}\pi_{k-2}$. Calling $\rho = \frac{p}{1-p}$, for $0 \leq j \leq k-1$ we should have:

$$\pi_j = \rho^j \pi_0$$

The last equation to be satisfied is: $\sum_{j=0}^{k-1} \pi_j = 1$. Thus we should have:

$$\sum_{j=0}^{k-1} \pi_j = \sum_{j=0}^{k-1} [\rho^j \pi_0] = 1$$

So we should have the following as the steady state distribution:

$$(1) \quad \pi_0 = \frac{1}{\sum_{j=0}^{k-1} \rho^j}$$

$$(2) \quad \pi_j = \rho^j \pi_0, \text{ for } 0 \leq j \leq k-1$$

When $\rho = 1$, looking at the steady state probability distribution equations or the set of equations given in (1) and (2), we will find that the steady state distribution is uniform among the different states:

$$\pi_j = \frac{1}{k}, \text{ for } 0 \leq j \leq k-1$$

b) For $\rho = 1/2$,

$$\sum_{j=0}^{k-1} \rho^j = 2(1 - \frac{1}{2^k})$$

. Thus,

$$\pi_0 = \frac{1}{2(1 - \frac{1}{2^k})}$$

For $\rho = 1$,

$$\sum_{j=0}^{k-1} \rho^j = k$$

. Thus,

$$\pi_0 = \frac{1}{k}$$

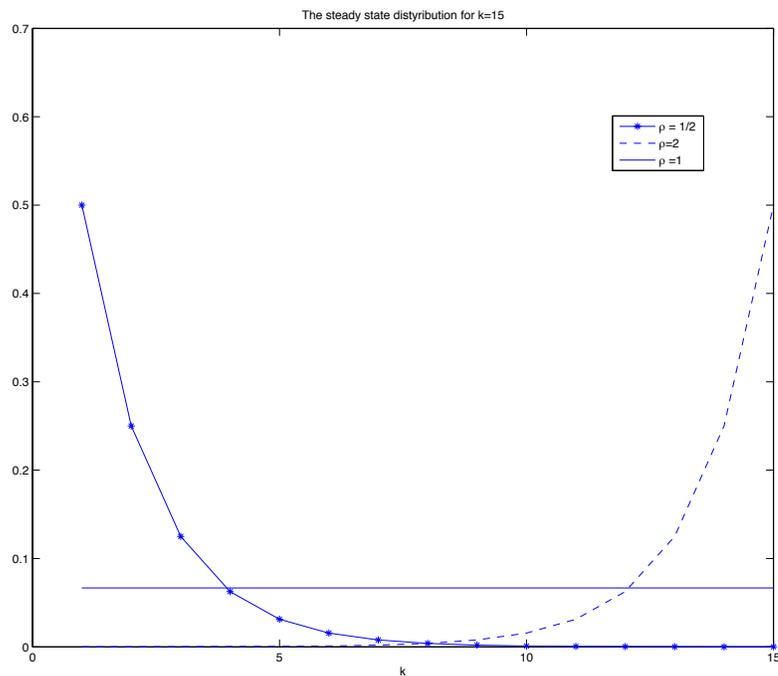
For $\rho = 2$,

$$\sum_{j=0}^{k-1} \rho^j = 2^k - 1$$

Thus,

$$\pi_0 = \frac{1}{2^k - 1}$$

And for all the cases, $\pi_j = \rho^j \pi_0$, $0 \leq j \leq k - 1$.



As you observe ρ can be considered as a measure of flow among the states in the Markov chain. When $\rho < 1$, the flow is towards left and the state 0 has the highest probability. When $\rho > 1$, the flow is towards right and the state $k - 1$ has the highest probability. When $\rho = 1$, there is no flow in any specific direction and the steady state distribution is uniform.

c) If $\rho \neq 1$, we have:

$$\pi_0 = \frac{1 - \rho}{1 - \rho^k}$$

$$\pi_{k-1} = \rho^{k-1} \frac{1 - \rho}{1 - \rho^k}$$

So if $\rho < 1$, as $k \rightarrow \infty$, $\pi_0 \rightarrow 1 - \rho$ and $\pi_{k-1} \rightarrow 0$.

And if $\rho > 1$, as $k \rightarrow \infty$, $\pi_0 \rightarrow 0$ and $\pi_{k-1} \rightarrow 1 - 1/\rho$.

For $\rho = 1$, the steady state distribution is uniform and $\pi_0 = \pi_{k-1} = 1/k \rightarrow 0$.

For $\rho < 1$, the PMF drops off geometrically with ρ , the process returns to the zero state often and rarely wanders off to the higher states. Taking the limit, $\lim_{k \rightarrow \infty} \pi_0 = (1 - \rho)$, $\lim_{k \rightarrow \infty} \pi_{k-1} = 0$. For $\rho = 1$, all probabilities are equal, and $\lim_{k \rightarrow \infty} \pi_0 = 0$, $\lim_{k \rightarrow \infty} \pi_{k-1} = 0$. For $\rho > 1$, the probabilities increase with i and as k gets larger, it is increasingly unlikely to be in state 0. Taking the limits, $\lim_{k \rightarrow \infty} \pi_0 = (1 - \rho)$, $\lim_{k \rightarrow \infty} \pi_{k-1} = 0$.

Solution to Exercise 3.10:

a) The transition matrix of the first graph is:

$$[P] = \begin{bmatrix} 0 & p & 0 & 1-p \\ 1-p & 0 & p & 0 \\ 0 & 1-p & 0 & p \\ p & 0 & 1-p & 0 \end{bmatrix}$$

The steady state distribution will satisfy the following equations:

$$\begin{aligned} \pi_1 &= (1-p)\pi_2 + p\pi_4 \\ \pi_2 &= (1-p)\pi_3 + p\pi_1 \\ \pi_3 &= (1-p)\pi_4 + p\pi_2 \\ \pi_4 &= (1-p)\pi_1 + p\pi_3 \\ 1 &= \pi_1 + \pi_2 + \pi_3 + \pi_4 \end{aligned}$$

Solving these set of equations, we get:

$$\pi_1 = \pi_2 = \pi_3 = \pi_4 = 1/4$$

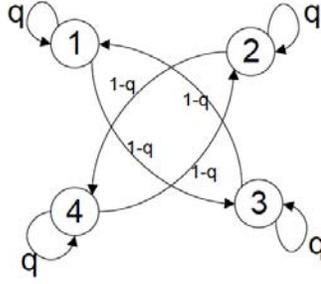
This was almost obvious considering the symmetry of the graph and equations.

b)

$$[P]^2 = \begin{bmatrix} 2p(1-p) & 0 & p^2 + (1-p)^2 & 0 \\ 0 & 2p(1-p) & 0 & p^2 + (1-p)^2 \\ p^2 + (1-p)^2 & 0 & 2p(1-p) & 0 \\ 0 & p^2 + (1-p)^2 & 0 & 2p(1-p) \end{bmatrix}$$

Let's define $q = 2p(1-p)$. Then the corresponding Markov chain to $[P_1]^2$ is:

You can observe that as the original Markov chain is periodic with period $d = 2$, $[P]^d$ partitions the Markov chain into d disjoint recurrent classes.



Since, $[P]^2$ has two recurrent classes, there are two independent steady state distributions for this Markov chain.

$$\pi_1 = [1/2, 0, 1/2, 0]$$

$$\pi_2 = [0, 1/2, 0, 1/2]$$

Any linear combination of these steady state distributions is also a steady state distribution of this Markov chain.

c) If the initial state of the Markov chain is 1 or 3, i.e., $X_0 \in \{1, 3\}$, the steady state distribution at time $2n$ as $n \rightarrow \infty$, is $[1/2, 0, 1/2, 0]$. If the initial state of the Markov chain is 2 or 4, i.e., $X_0 \in \{2, 4\}$, the steady state distribution at time $2n$ as $n \rightarrow \infty$, is $[0, 1/2, 0, 1/2]$. Thus we will have:

$$\lim_{n \rightarrow \infty} [P]^{2n} = \begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{bmatrix}$$

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