# Solutions to Homework 2

6.262 Discrete Stochastic Processes
MIT, Spring 2011

**Solution to Exercise 1.10:**

**a)** We know that $Z(\omega) = X(\omega) + Y(\omega)$ for each $\omega \in \Omega$.

$$\begin{aligned}\Pr(Z(\omega) = \pm\infty) &= \Pr\{\omega; Z(\omega) = +\infty \text{ or } Z(\omega) = -\omega\} \\ &= \Pr\{\omega; Z(\omega) = +\infty\} + \Pr\{\omega; Z(\omega) = -\infty\}\end{aligned}$$

$$\begin{aligned}\Pr\{\omega; Z(\omega) = +\infty\} &= \Pr\{\omega; X(\omega) = \infty \text{ or } Y(\omega) = \infty\} \\ &\le \Pr\{\omega; X(\omega) = \infty\} + \Pr\{\omega; Y(\omega) = \infty\} \\ &= 0 + 0 = 0.\end{aligned}$$

We know that $X$ is a random variable and based on the definition of random variables, we know that $\Pr\{\omega; X(\omega) = \infty\} = 0$.

Similarly, it can be proved that $\Pr\{\omega; Z(\omega) = -\infty\} = 0$. Thus, $\Pr\{\omega; Z(\omega) = \pm\infty\} = 0$.

**Solution to Exercise 1.17:**

**a)**

Since $Y$ is integer-valued, $F_Y(y)$ and $F_Y^c$ are constant between integer values. Thus, $\int_0^\infty F_Y(y)\,dy = \sum_{y=0}^\infty F_Y^c(y)$.

$$\begin{aligned}\mathbb{E}[Y] &= \sum_{y=0}^{+\infty} F_Y^c(y) \\ &= \sum_{y=0}^{+\infty} \frac{2}{(y+1)(y+2)} \\ &= \sum_{y=0}^{+\infty} \frac{2}{(y+1)} - \frac{2}{(y+2)} \\ &= \frac{2}{1} = 2\end{aligned}$$

**b)** We can compute the PMF :

$$
\begin{aligned}
p_Y(y) &= F_Y(y) - F_Y(y-1) \\
&= 1 - \frac{2}{(y+1)(y+2)} - (1 - \frac{2}{y(y+1)}) \\
&= \frac{4}{y(y+1)(y+2)}, \forall y > 0.
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[Y] &= \sum_{y=1}^{+\infty} y.p_Y(y) \\
&= \sum_{y=1}^{+\infty} \frac{4}{(y+1)(y+2)} \\
&= \sum_{y=1}^{+\infty} (\frac{4}{(y+1)} - \frac{4}{(y+2)}) = \frac{4}{2} = 2.
\end{aligned}
$$

**c)** Condition on the event $[Y = y]$, the rv $X$ has a uniform distribution over the interval $[1, y]$. So, $\mathbb{E}[X|Y = y] = (1 + y)/2$.

$$
\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y = y]] = \mathbb{E}[\frac{1+y}{2}] = \frac{1 + \mathbb{E}[y]}{2} = 3/2
$$

$$
\begin{aligned}
p_X(x) &= \sum_{y=1}^{y=\infty} p_{X|Y}(x|y)p_Y(y) \\
&= \sum_{y=x}^{y=\infty} \frac{4}{y^2(y+1)(y+2)} \\
&= \sum_{y=x}^{y=\infty} \frac{-3}{y} + \frac{2}{y^2} + \frac{4}{y+1} - \frac{1}{y+2}
\end{aligned}
$$

It's too complicated to calculate this term.

**d)** Condition on the event $[Y = y]$, the rv $Z$ has a uniform distribution over the interval $[1, y^2]$. So, $\mathbb{E}[X|Y = y] = (1 + y^2)/2$.

$$
\mathbb{E}[Z] = [\frac{1+y^2}{2}] = \frac{1}{2} + \frac{1}{2} \sum_{y=1}^{\infty} \frac{4y}{(y+1)(y+2)} = \infty
$$

**Solution to Exercise 1.31:**

**a)**

We know that:

$$|\int_{-\infty}^{0} e^{rx}\,\mathrm{d}F(x) \leq \int_{-\infty}^{0} |e^{rx}|\,\mathrm{d}F(x)$$

And if $r \geq 0$, for $x < 0$, $|e^{rx}| \leq 1$. Thus,

$$|\int_{-\infty}^{0} e^{rx}\,\mathrm{d}F(x)| \leq \int_{-\infty}^{0} \mathrm{d}F(x) \leq 1$$

Similarly, when $r \leq 0$, $|e^{rx}| \leq 1$ for $x \geq 0$

$$|\int_{0}^{\infty} e^{rx}\,\mathrm{d}F(x)| \leq \int_{0}^{\infty} |e^{rx}|\,\mathrm{d}F(x) \leq \int_{0}^{\infty} \mathrm{d}F(x) \leq 1$$

**b)** We know that for all $x \geq 0$, for each $0 \leq r \leq r_1$, $e^{rx} \leq e^{r_1 x}$. Thus,

$$\int_{0}^{\infty} e^{rx}\,\mathrm{d}F(x) \leq \int_{0}^{\infty} e^{r_1 x}\,\mathrm{d}F(x) < \infty$$

**c)** We know that for all $x \leq 0$, for each $r_2 \leq r \leq 0$, $e^{rx} \leq e^{r_2 x}$. Thus,

$$\int_{-\infty}^{0} e^{rx}\,\mathrm{d}F(x) \leq \int_{-\infty}^{0} e^{r_2 x}\,\mathrm{d}F(x) < \infty$$

**d)** We know the value of $g_X(r)$ for $r = 0$:

$$g_X(0) = \int_{-\infty}^{0} dF(x) + \int_{0}^{\infty} dF(x) = 1$$

So for $r = 0$, the moment generating function exists. We also know that the first integral exists for all $r \geq 0$. And if the second integral exists for a given $r_1 > 0$, it exists for all $0 \leq r \leq r_1$. So if $g_X(r)$ exists for some $r_1 \geq 0$, it exists for all $0 \leq r \leq r_1$. Similarly, we can prove that if $g_X(r)$ exists for some $r_2 \leq 0$, it exists for all $r_2 \leq r \leq 0$. So the interval in which $g_X(r)$ exists is from some $r_2 \leq 0$ to some $r_1 \geq 0$.

**e)** We note immediately that with $f_X(x) = e^{-x}$ for $x \geq 0$, $g_X(1) = \infty$. With $f_X(x) = (ax^{-2})e^{-x}$ for $x \geq 1$ and 0 otherwise, it is clear that $g_X(1) < \infty$. Here $a$ is taken to be such that $\int_1^{\infty}(ax^{-2})e^{-x}\,\mathrm{d}x = 1$.

**Solution to Exercise 1.33:**

Given that $X_i$'s are IID, we have $\mathbb{E}[S_n] = n\mathbb{E}[X] = n\delta$ and $Var(S_n) = nVar(X) = n\sigma^2$ where $Var(X) = \sigma^2 = \delta(1-\delta)$. Since $Var(X) < \infty$, the CLT implies that the normalized version of $S_n$, defined as $Y_n = (S_n - n\mathbb{E}[X])/(\sqrt{n}\sigma)$ will tend to a normalized Gaussian distribution as $n \to \infty$, so we can use the integral in (1.81). First let us see what is happening and then we will verify the results.

**a)** The standard deviation of $S_n$ is increasing in $\sqrt{n}$ while the interval of the summation is a fixed interval about the mean. Thus, the probability distribution becomes flatter and more spread out as $n$ increases, and the probability of interest tends to 0. Analytically, let $Y_n = (S_n - n\mathbb{E}[X])/(\sqrt{n}\sigma)$, then,

$$(1) \qquad \sum_{i:n\delta - m \le i \le n\delta + m} \Pr\{S_n = i\} \sim F_{S_n}(n\delta + m) - F_{S_n}(n\delta - m) = F_{Y_n}(\frac{m}{\sqrt{n}\sigma}) - F_{Y_n}(-\frac{m}{\sqrt{n}\sigma})$$

For large $n$, $Y_n$ becomes a standard normal,

$$(2) \qquad \lim_{n\to\infty} \sum_{i:n\delta - m \le i \le n\delta + m} \Pr\{S_n = i\} = \lim_{n\to\infty}(F_{Y_n}(\frac{m}{\sqrt{n}\sigma}) - F_{Y_n}(-\frac{m}{\sqrt{n}\sigma})) = 0$$

Comment: To be rigorous, which is not necessary here, the limit in (2) is taken as follows. We need to show that $F_{Y_n}(y_n) \to \Phi(y)$ ($\Phi$ is the cumulative gaussian distribution), where the sequence $y_n \to y$ and the functions $F_{Y_n} \to \Phi$ (pointwise). Consider an $\epsilon > 0$ and pick $N$ large enough so that for all $n \ge N$ we have $y - \epsilon \le y_n \le y + \epsilon$. Then we get, $F_{Y_n}(y - \epsilon) \le F_{Y_n}(y_n) \le F_{Y_n}(y + \epsilon)$, $\forall n > N$ (since $F_{Y_n}$ is non-decreasing). Take the limit $n \to \infty$ which gives $\Phi(y - \epsilon) \le \lim_{n\to\infty} F_{Y_n}(y_n) \le \Phi(y + \epsilon)$ (by CLT). Now, take the limit $\epsilon \to \infty$ and since $\Phi(.)$ is a continuous function, both sides of the inequality converge to the same value $\Phi(y)$ which completes the proof.

**b)** The summation here is over all terms below the mean plus the terms which exceed the mean by at most $m$. As $n \to \infty$, the normalized distribution (the distribution of $Y_n$) becomes Gaussian and the integral up to the mean becomes 1/2. Analytically,

$$\lim_{n\to\infty} \sum_{i:0 \le i \le n\delta + m} \Pr\{S_n = i\} = \lim_{n\to\infty}(F_{Y_n}(\frac{m}{\sqrt{n}\sigma}) - F_{Y_n}(\frac{-n\delta}{\sqrt{n}\sigma})) = 1/2$$

**c)** The interval of summation is increasing with $n$ while the standard deviation is increasing with $\sqrt{n}$. Thus, in the limit, the probability of interest will include all the probability mass.

$$\lim_{n\to\infty} \sum_{i:n(\delta - 1/m) \le i \le n(\delta + 1/m)} \Pr\{S_n = i\} = \lim_{n\to\infty}(F_{Y_n}(\frac{n}{m\sqrt{n}\sigma}) - F_{Y_n}(\frac{-n}{m\sqrt{n}\sigma})) = 1$$

## Solution to Exercise 1.38:

We know that $S_n$ is a r.v. with mean 0 and variance $n\sigma^2$. According to CLT, both $S_n/\sigma\sqrt{n}$ and $S_{2n}/\sigma\sqrt{2n}$ converge in distribution to normal distribution with mean 0 and variance 1. But this does not imply the convergence of $S_n/\sigma\sqrt{n} - S_{2n}/\sigma\sqrt{2n}$.

$$\frac{S_n}{\sigma\sqrt{n}} - \frac{S_{2n}}{\sigma\sqrt{2n}} = \frac{\sqrt{2}S_n - S_{2n}}{\sigma\sqrt{2n}}$$
$$= \frac{\sqrt{2}\sum_{i=1}^{n} X_i - \sum_{i=1}^{2n} X_i}{\sigma\sqrt{2n}}$$
$$= (\frac{\sqrt{2}-1}{\sqrt{2}})\frac{\sum_{i=1}^{n} X_i}{\sigma\sqrt{n}} - \frac{1}{\sqrt{2}}\frac{\sum_{i=n+1}^{2n} X_i}{\sigma\sqrt{n}}$$

Independency of $X_i$'s imply the independency of $S_n = \sum_{i=1}^{n} X_i$ and $S'_n = \sum_{i=n+1}^{2n} X_i$. Both $S_n/(\sigma\sqrt{n})$ and $S'_n/(\sigma\sqrt{n})$ converge in distribution to zero mean, unit variance normal distribution and they are independent. Thus, $(\frac{\sqrt{2}-1}{\sqrt{2}})\frac{S_n}{\sigma\sqrt{n}}$ converges to a normal r.v. with mean 0 and variance $3/2 - \sqrt{2}$ and is independent of $(\frac{1}{\sqrt{2}})\frac{S'_n}{\sigma\sqrt{n}}$ which converges in distribution to a normal r.v. with mean 0 and variance 1/2.

So, $\frac{S_n}{\sigma\sqrt{n}} - \frac{S_{2n}}{\sigma\sqrt{2n}}$ converges in distribution to a normal r.v with mean 0 and variance $2 - \sqrt{2}$. This means that $\frac{S_n}{\sigma\sqrt{n}} - \frac{S_{2n}}{\sigma\sqrt{2n}}$ does not converge to a constant and it might take different values with the described probability distribution.

**Solution to Exercise 1.42:**

**a)**

$$\bar{X} = 100$$

$$\sigma_X^2 = (10^{12} - 100)^2 \times 10^{-10} + (100+1)^2 \times (1 - 10^{-10})/2 + (100-1)^2 \times (1 - 10^{-10})/2$$
$$= 10^{14} - 2 \times 10^4 + 10^4 \approx 10^{14}.$$

Thus,

$$\bar{S_n} = 100n$$
$$\sigma_{S_n}^2 = n \times 10^{14}$$

**b)** This is the event that all $10^6$ trials result in $\pm 1$. That is, there are no occurrences of $10^{12}$. That is, there are no occurrences of $10^{12}$. Thus, $\Pr S_n \le 10^6 = (1 - 10^{-10})^{10^6}$.

**c)** From the union bound, the probability of one or more occurrences of the sample value $10^{12}$ out of $10^6$ trials is bounded by a sum over $10^6$ terms, each of which is $10^{-10}$, i.e., $1 - F_{S_n}(10^6) \le 10^{-4}$.

**d)** Conditional on no occurrences of $10^{12}$, $S_n$ simply has a binomial distribution. We know from the central limit theorem for the binomial case that $S_n$ will be approximately Gaussian with mean 0 and standard deviation $10^3$. Since one or more occurrences of $10^{12}$ occur only with probability $10^{-4}$, this can be neglected, so the distribution function is approximately Gaussian with 3 sigma points at $\pm 3 \times 10^3$.

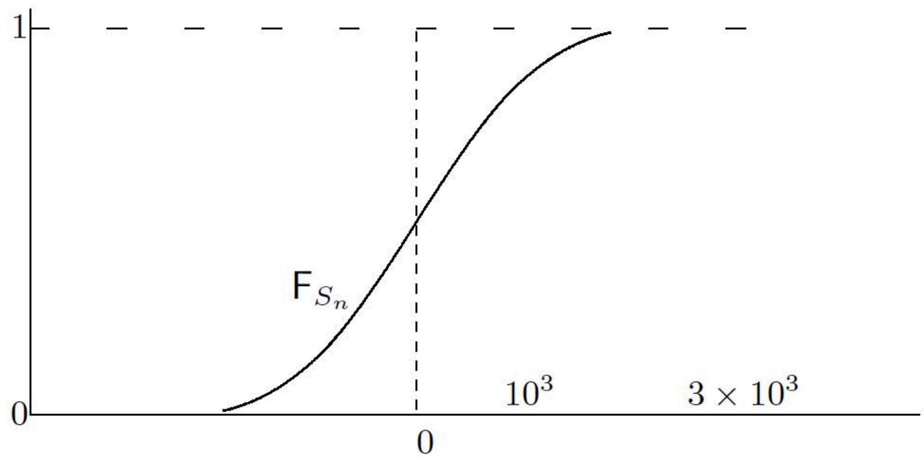


FIGURE 1. Distribution function of $S_n$ for $n = 10^6$

**e)** First, consider the PMF $p_B(j) =$ of the number $B = j$ of occurrences of the value $10^{12}$. We have

$$p_B\,(j) = \binom{10^{10}}{j} q^j (1-q)^{10^{10}-j}$$

where $q = 10^{-10}$.

$$p_B\,(0) = (1-q)^{10^{10}} = \exp(10^{10} \ln{[1-q]}) \approx \exp\left(-10^{10}q\right) = e^{-1}$$

$$p_B\,(1) = 10^{10} q (1-q)^{10^{10}-1} = (1-q)^{10^{10}-1} \approx e^{-1}$$

$$p_B\,(2) = \binom{10^{10}}{2} q^2 (1-q)^{10^{10}-2} = \frac{10^{10} \times (10^{10}-1)}{2} (10^{-10})^2 (1-10^{-10})^{10^{10}-2} \approx \frac{1}{2} e^{-1}$$

Thus,

$$\Pr\left(B \le 2\right) \approx 2.5 e^{-1}$$

Conditional on $\{B = j\}$, $S_n$ will be approximately Gaussian with mean $10^{12}j$ and standard deviation of $10^5$. Thus $F_{S_n}(x)$ rises from 0 to $e^{-1}$ over a range of $x$ from about $-3 \times 10^5$ to $+3 \times 10^5$. It then stays virtually constant up to about $x = 10^{12} - 3 \times 10^5$. It rises to $2e^{-1}$ by about $x = 10^{12} + 3 \times 10^5$. It stays virtually constant up to about $2 \times 10^{12} - 3 \times 10^5$ and rises to $2.5e^{-1}$ by about $2 \times 10^{12} + 3 \times 10^5$. When we sketch this, the rises in $F_{S_n}(x)$ for $n = 10^{10}$ over a width of about $6 \times 10^5$ look essentially vertical on a scale of $2 \times 10^{12}$, rising from 0 to $e^{-1}$ at 0, from $1/e$ to $2/e$ at $10^{12}$ and from $2/e$ to $2.5/e$ at $2 \times 10^{12}$. There are smaller steps at larger values, but they would scarcely show up on this sketch.

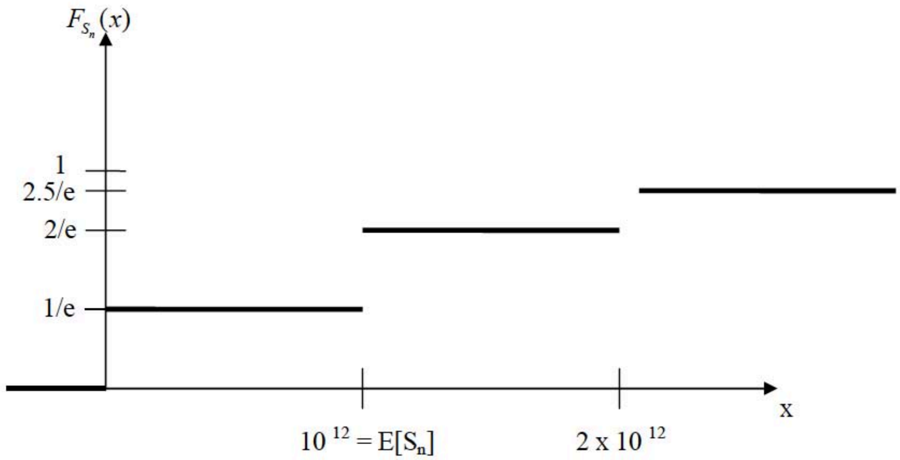


FIGURE 2. Distribution function of $S_n$ for $n = 10^{10}$

**f)** It can be seen that for this peculiar rv, $S_n/n$ is not concentrated around its mean even for $n = 10^{10}$ and $S_n/\sqrt{(n)}$ does not look Gaussian even for $n = 10^{10}$. For this particular distribution, $n$ has to be so large that $B$, the number of occurrences of $10^{12}$, is large, and this requires $n \gg 10^{10}$. This illustrates a common weakness of limit theorems. They say what happens as a parameter ($n$ in this case) becomes sufficiently large, but it takes extra work to see how large that is.

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