6.254 : Game Theory with Engineering Applications
Lecture 14: Nash Bargaining Solution

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Outline

- Rubinstein Bargaining Model with Alternating Offers
- Nash Bargaining Solution
- Relation of Axiomatic and Strategic Model

- Reference:
- Osborne and Rubinstein, Bargaining and Markets.

Introduction

- In this lecture, we discuss an axiomatic approach to the bargaining problem.
- In particular, we introduce the Nash bargaining solution and study the relation between the axiomatic and strategic (noncooperative) models.
- As we have seen in the last lecture, the Rubinstein bargaining model allows two players to offer alternating proposals indefinitely, and it assumes that future payoffs of players 1 and 2 are discounted by $\delta_1, \delta_2 \in (0, 1)$.

Rubinstein Bargaining Model with Alternating Offers

- We showed that the following stationary strategy profile is a subgame perfect equilibrium for this game.
  - Player 1 proposes x_1^* and accepts offer y if, and only if, $y \geq y_1^*$.
  - Player 2 proposes y_2^* and accepts offer x if, and only if, $x \geq x_2^*$,

  where

$$x_1^* = \frac{1-\delta_2}{1-\delta_1\delta_2}, \qquad y_1^* = \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2},$$

$$x_2^* = \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}, \qquad y_2^* = \frac{1-\delta_1}{1-\delta_1\delta_2}.$$

- Clearly, an agreement is reached immediately for any values of δ_1 and δ_2.
- To gain more insight into the resulting allocation, assume for simplicity that $\delta_1 = \delta_2$. Then, we have
  - If 1 moves first, the division will be $\left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right)$.
  - If 2 moves first, the division will be $\left(\frac{\delta}{1+\delta}, \frac{1}{1+\delta}\right)$.

Rubinstein Bargaining Model with Alternating Offers

- The first mover's advantage (FMA) is clearly related to the impatience of the players (i.e., related to the discount factor δ):
  - If $\delta \longrightarrow 1$, the FMA disappears and the outcome tends to $(\frac{1}{2}, \frac{1}{2})$.
  - If $\delta \longrightarrow 0$, the FMA dominates and the outcome tends to $(1, 0)$.
- More interestingly, let's assume the discount factor is derived from some interest rates r_1 and r_2.

$$\delta_1 = e^{-r_1 \Delta t}, \qquad \delta_2 = e^{-r_2 \Delta t}$$

- These equations represent a continuous-time approximation of interest rates. It is equivalent to interest rates for very small periods of time Δt: $e^{-r_i \Delta t} \simeq \frac{1}{1+r_i \Delta t}$.
- Taking $\Delta t \longrightarrow 0$, we get rid of the first mover's advantage.

$$\lim_{\Delta t \to 0} x_1^* = \lim_{\Delta t \to 0} \frac{1-\delta_2}{1-\delta_1 \delta_2} = \lim_{\Delta t \to 0} \frac{1-e^{-r_2 \Delta t}}{1-e^{-(r_1+r_2)\Delta t}} = \frac{r_2}{r_1+r_2}.$$

Alternative Bargaining Model: Nash's Axiomatic Model

- Bargaining problems represent situations in which:
  - There is a conflict of interest about agreements.
  - Individuals have the possibility of concluding a mutually beneficial agreement.
  - No agreement may be imposed on any individual without his approval.
- The strategic or noncooperative model involves explicitly modeling the bargaining process (i.e., the game form).
- We will next adopt an axiomatic approach, which involves abstracting away the details of the process of bargaining and considers only the set of outcomes or agreements that satisfy "reasonable" properties.
- This approach was proposed by Nash in his 1950 paper, where he states "One states as axioms several properties that would seem natural for the solution to have and then one discovers that axioms actually determine the solution uniquely."
- The first question to answer is: What are some reasonable axioms?
- To gain more insight, let us start with a simple example.

Nash's Axiomatic Model

Example

Suppose 2 players must split one unit of a good. If no agreement is reached, then players do not receive anything. Preferences are identical. We then expect:

- *Players to agree (Efficiency)*
- *Each to obtain half (Symmetry)*

- We next consider a more general scenario.
- We use X to denote set of possible agreements and D to denote the disagreement outcome.
  - As an example we may have

$$X = \{(x_1, x_2) | x_1 + x_2 = 1, x_i \geq 0\}, \; D = (0, 0).$$

Nash's Axiomatic Model

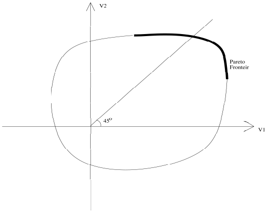
- We assume that each player i has preferences, represented by a utility function u_i over $X \cup \{D\}$. We denote the set of possible payoffs by set U defined by

$$\begin{aligned} U &= \{(v_1, v_2) \mid u_1(x) = v_1, u_2(x) = v_2 \text{ for some } x \in X\} \\ d &= (u_1(D), u_2(D)) \end{aligned}$$

- A bargaining problem is a pair (U, d) where $U \subset \mathbb{R}^2$ and $d \in U$. We assume that
  - U is a convex and compact set.
  - There exists some $v \in U$ such that $v > d$ (i.e., $v_i > d_i$ for all i).
- We denote the set of all possible bargaining problems by $\mathcal{B}$.
- A bargaining solution is a function $f : \mathcal{B} \to U$.
- We will study bargaining solutions $f(\cdot)$ that satisfy a list of reasonable axioms.

Axioms

- Pareto Efficiency:
  - A bargaining solution $f(U,d)$ is Pareto efficient if there does not exist a $(v_1, v_2) \in U$ such that $v \geq f(U,d)$ and $v_i > f_i(U,d)$ for some i.
  - An inefficient outcome is unlikely, since it leaves space for renegotiation.



- Symmetry:
  - Let (U,d) be such that $(v_1, v_2) \in U$ if and only if $(v_2, v_1) \in U$ and $d_1 = d_2$. Then $f_1(U,d) = f_2(U,d)$.
  - If the players are indistinguishable, the agreement should not discriminate between them.

Axioms

- Invariance to Equivalent Payoff Representations
  - Given a bargaining problem (U, d), consider a different bargaining problem (U', d') for some $\alpha > 0, \beta$:

$$\begin{aligned} U' &= \{(\alpha_1 v_1 + \beta_1, \alpha_2 v_2 + \beta_2) \mid (v_1, v_2) \in U\} \\ d' &= (\alpha_1 d_1 + \beta_1, \alpha_2 d_2 + \beta_2) \end{aligned}$$

  - Then, $f_i(U', d') = \alpha_i f_i(U, d) + \beta_i$.
  - Utility functions are only representation of preferences over outcomes. A transformation of the utility function that maintains the some ordering over preferences (such as a linear transformation) should not alter the outcome of the bargaining process.
- Independence of Irrelevant Alternatives
  - Let (U, d) and $(U'd)$ be two bargaining problems such that $U' \subseteq U$. If $f(U, d) \in U'$, then $f(U', d) = f(U, d)$.

Nash Bargaining Solution

Definition

We say that a pair of payoffs (v_1^*, v_2^*) is a Nash bargaining solution if it solves the following optimization problem:

$$\begin{aligned} \max_{v_1, v_2} \quad & (v_1 - d_1)(v_2 - d_2) \\ \textit{subject to} \quad & (v_1, v_2) \in U \\ & (v_1, v_2) \geq (d_1, d_2) \end{aligned} \tag{1}$$

We use $f^N(U, d)$ to denote the Nash bargaining solution.

Remarks:

- *Existence of an optimal solution*: Since the set U is compact and the objective function of problem (1) is continuous, there exists an optimal solution for problem (1).
- *Uniqueness of the optimal solution*: The objective function of problem (1) is strictly quasi-concave. Therefore, problem (1) has a unique optimal solution.

Nash Bargaining Solution

Proposition

Nash bargaining solution $f^N(U, d)$ is the unique bargaining solution that satisfies the 4 axioms.

Proof: The proof has 2 steps. We first prove that Nash bargaining solution satisfies the 4 axioms. We then show that if a bargaining solution satisfies the 4 axioms, it must be equal to $f^N(U, d)$.

Step 1:

- Pareto efficiency: This follows immediately from the fact that the objective function of problem (1) is increasing in v_1 and v_2.
- Symmetry: Assume that $d_1 = d_2$. Let $v^* = (v_1^*, v_2^*) = f^N(U, d)$ be the Nash bargaining solution. Then, it can be seen that (v_2^*, v_1^*) is also an optimal solution of (1). By the uniqueness of the optimal solution, we must have $v_1^* = v_2^*$, i.e, $f_1^N(U, d) = f_2^N(U, d)$.

Nash Bargaining Solution

- Independence of irrelevant alternatives: Let $U' \subseteq U$. From the optimization problem characterization of the Nash bargaining solution, it follows that the objective function value at the solution $f^N(U,d)$ is greater than or equal to that at $f^N(U',d)$. If $f^N(U,d) \in U'$, then the objective function values must be equal, i.e $f^N(U,d)$ is optimal for U' and by uniqueness of the solution $f^N(U,d) = f^N(U',d)$.
- Invariance to equivalent payoff representations: By definition, $f(U',d')$ is an optimal solution of the problem

$$\max_{v_1,v_2}(v_1 - \alpha_1 d_1 - \beta_1)(v_2 - \alpha_2 d_2 - \beta_2)$$
$$\text{s.t } (v_1,v_2) \in U'$$

Performing the change of variables $v_1' = \alpha_1 v_1 + \beta_1$ $v_2' = \alpha_2 v_2 + \beta_2$, it follows immediately that $f_i^N(U',d') = \alpha_i f_i^N(U,d) + \beta_i$ for $i = 1,2$.

Nash Bargaining Solution

Step 2: Let $f(U, d)$ be a bargaining solution satisfying the 4 axioms. We prove that $f(U, d) = f^N(U, d)$.

- Let $z = f^N(U, d)$, and define the set

$$U' = \{\alpha' v + \beta | v \in U; \alpha' z + \beta = (1/2, 1/2)'; \alpha' d + \beta = (0, 0)'\},$$

  i.e., we map the point z to $(1/2, 1/2)$ and the point d to (0,0).
  Since $f(U, d)$ and $f^N(U, d)$ both satisfy axiom 3 (invariance to equivalent payoff representations), we have $f(U, d) = f^N(U, d)$ if and only if $f(U', 0) = f^N(U', 0) = (1/2, 1/2)$. Hence, to establish the desired claim, it is sufficient to prove that $f(U', 0) = (1/2, 1/2)$.

- Let us show that there is no $v \in U'$ such that $v_1 + v_2 > 1$:
  Assume that there is a $v \in U'$ such that $v_1 + v_2 > 1$. Let $t = (1 - \lambda)(1/2, 1/2) + \lambda(v_1, v_2)$ for some $\lambda \in (0, 1)$. Since U' is convex, we have $t \in U'$. We can choose λ sufficiently small so that $t_1 t_2 > 1/4 = f^N(U', 0)$, but this contradicts the optimality of $f^N(U', 0)$, showing that for all $v \in U'$, we have $v_1 + v_2 \leq 1$.

Nash Bargaining Solution

- Since U' is bounded, we can find a rectangle U'' symmetric with respect to the line $v_1 = v_2$, such that $U' \subseteq U''$ and $(1/2, 1/2)$ is on the boundary of U''.
- By Axioms 1 and 2, $f(U'', 0) = (1/2, 1/2)$.
- By Axiom 4, since $U' \subseteq U''$, we have $f(U', 0) = (1/2, 1/2)$, completing the proof.

Example: Dividing a Dollar

Let $X = \{(x_1, x_2) | x_1 \geq 0,\ x_1 + x_2 = 1\}$, $D = (0, 0)$. We define the set of payoffs as

$$U = \{(v_1, v_2) | (v_1, v_2) = (u_1(x_1), u_2(x_2)),\ (x_1, x_2) \in X\},$$

and $d = (u_1(0), u_2(0))$. We study the Nash bargaining solution for two cases:

- Case 1. $u_1 = u_2 = u$ (u: concave and $u(0) = 0$): symmetric bargaining problem. Hence $f^N(U, d) = (1/2, 1/2)$: the dollar is shared equally. The Nash bargaining solution is the optimal solution of the following problem

  $$\max_{0 \leq z \leq 1} v_1(z) v_2(1 - z) = u(z) u(1 - z).$$

  We denote the optimal solution of this problem by z_u. By the first order optimality conditions, we have

  $$u'(z) u(1 - z) = u(z) u'(1 - z),$$

  implying that $\frac{u'(z_u)}{u(z_u)} = \frac{u'(1 - z_u)}{u(1 - z_u)}$.

Example: Dividing a Dollar

- Case 2. Player 2 is more risk averse, i.e., $v_1 = u$, $v_2 = h \circ u$, where $h : \mathbb{R} \to \mathbb{R}$ is an increasing concave function with $h(0) = 0$. The Nash bargaining solution is the optimal solution of the following problem

$$\max_{0 \leq z \leq 1} v_1(z) v_2(1-z) = u(z) h(u(1-z))$$

We denote the optimal solution of this problem by z_v By the first order optimality conditions, we have

$$u'(z) h(u(1-z)) = u(z) h'(u(1-z)) u'(1-z),$$

implying that $\frac{u'(z_v)}{u(z_v)} = \frac{h'(u(1-z_v)) u'(1-z_v)}{h(u(1-z_v))}$.
Since h is a concave increasing function and $h(0) = 0$, we have for

$$h'(t) \leq \frac{h(t)}{t} \qquad \text{for all } t \geq 0.$$

This implies that

$$\frac{u'(z_v)}{u(z_v)} \leq \frac{u'(1-z_v)}{u(1-z_v)},$$

and therefore $z_u \leq z_v$. the preceding analysis shows that when player 2 is more risk averse, player 1's share increases.

Relation of Axiomatic Model to Extensive Game Model

- We next consider variations on Rubinstein's bargaining model with alternating offers and compare the resulting allocations with the Nash bargaining solution.
- To compare, we need a disagreement outcome.
- We do this by considering two different scenarios:
  - Availability of outside options
  - Risk of breakdown

Outside Options Model

- When responding to an offer, player 2 may pursue an outside option $(0, d_2)$ with $d_2 \geq 0$.
- We have the following result for this game.
  - If $d_2 \leq x_2^*$ then the strategy pair of Rubinstein's bargaining model is the unique SPE.
  - If $d_2 > x_2^*$ then the game has a unique SPE in which:
    Player 1 always proposes $(1 - d_2, d_2)$ and accepts a proposal y if and only if $y_1 \geq \delta_1(1 - d_2)$.
    Player 2 always proposes $(\delta_1(1 - d_2), 1 - \delta_1(1 - d_2))$ and accepts a proposal x if and only if $x_2 \geq d_2$.
- Intuition: A player's outside option has value only if it is worth more than her equilibrium payoff in its absence.
- Compare with the Nash bargaining solution x_N given by

$$x_N = \arg\max_{x \geq 0} (x - d_1)(1 - x - d_2) = 1/2 + 1/2(d_1 - d_2).$$

This allocation is different from the equilibrium allocation in which the outside option $(0, d_2)$ only has an effect if $d_2 > x_2^*$.

Bargaining with the Risk of Breakdown

- In this model, there is an exogenous probability α of breaking down.
- We can assume without loss of generality that $\delta \rightarrow 1$, since the possibility of a breakdown puts pressure to reach an agreement.
- It can be seen that this game has a unique SPE in which,
  Player 1 proposes $\hat{x}$ and accepts an offer y if and only if $y_1 \geq \hat{y}_1$.
  Player 2 proposes $\hat{y}$ and accepts an offer x if and only if $x_1 \geq \hat{x}_1$,
  where

$$\hat{x}_1 = \frac{1 - d_2 + (1-\alpha)d_1}{2-\alpha}, \qquad \hat{y}_1 = \frac{(1-\alpha)(1-d_2) + d_1}{2-\alpha}$$

- Letting $\alpha \rightarrow 0$, we have $\hat{x}_1 \rightarrow 1/2 + 1/2(d_1 - d_2)$, which coincides with the Nash bargaining solution.
- That is, the variant of the bargaining game with alternating offers with exogenous probabilistic breakdown and Nash's axiomatic model, though built entirely of different components, yield the same outcome.