LECTURE 17

LECTURE OUTLINE

- Review of cutting plane method
- Simplicial decomposition
- Duality between cutting plane and simplicial decomposition

CUTTING PLANE METHOD

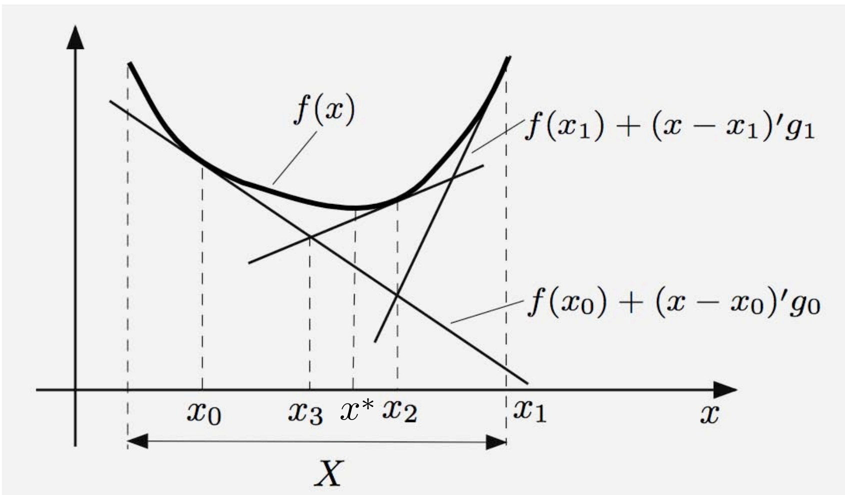
- Start with any $x_0 \in X$. For $k \geq 0$, set

$$x_{k+1} \in \arg\min_{x \in X} F_k(x),$$

where

$$F_k(x) = \max\Big\{f(x_0)+(x-x_0)'g_0, \ldots, f(x_k)+(x-x_k)'g_k\Big\}$$

and g_i is a subgradient of f at x_i.



- We have $F_k(x) \leq f(x)$ for all x, and $F_k(x_{k+1})$ increases monotonically with k.

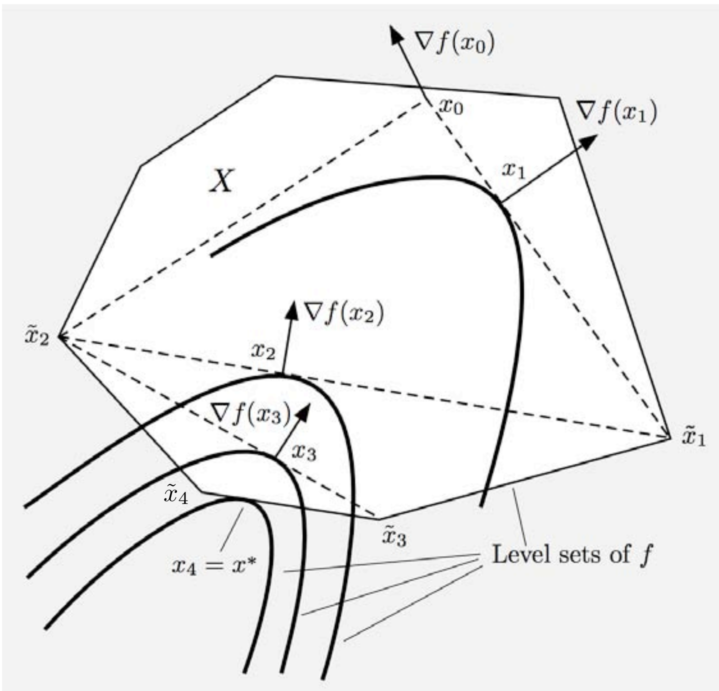
- These imply that all limit points of x_k are optimal.

BASIC SIMPLICIAL DECOMPOSITION

- Minimize a *differentiable* convex $f : \Re^n \mapsto \Re$ over *bounded polyhedral constraint set* X.

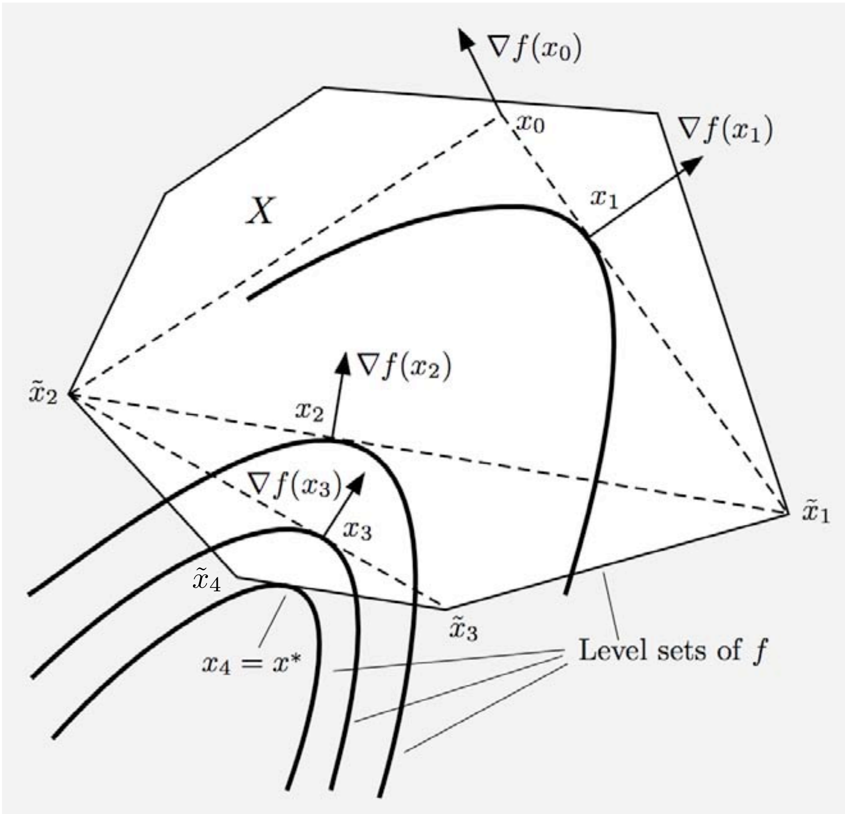
- **Approximate** X with a simpler inner approximating polyhedral set.

- Construct a more refined problem by solving a **linear** minimization over the original constraint.



- The method is appealing under two conditions:
  - Minimizing f over the convex hull of a relative small number of extreme points is much simpler than minimizing f over X.
  - Minimizing a linear function over X is much simpler than minimizing f over X.

SIMPLICIAL DECOMPOSITION METHOD



- Given current iterate x_k, and finite set $X_k \subset X$ (initially $x_0 \in X$, $X_0 = \{x_0\}$).

- Let $\tilde{x}_{k+1}$ be extreme point of X that solves

$$\begin{aligned} &\text{minimize} \quad \nabla f(x_k)'(x - x_k) \\ &\text{subject to} \quad x \in X \end{aligned}$$

and add $\tilde{x}_{k+1}$ to X_k: $X_{k+1} = \{\tilde{x}_{k+1}\} \cup X_k$.

- Generate x_{k+1} as optimal solution of

$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \quad x \in \text{conv}(X_{k+1}). \end{aligned}$$

CONVERGENCE

- There are two possibilities for $\tilde{x}_{k+1}$:

  (a) We have

  $$0 \le \nabla f(x_k)'(\tilde{x}_{k+1}-x_k) = \min_{x \in X} \nabla f(x_k)'(x-x_k)$$

  Then x_k minimizes f over X (satisfies the optimality condition)

  (b) We have

  $$0 > \nabla f(x_k)'(\tilde{x}_{k+1} - x_k)$$

  Then $\tilde{x}_{k+1} \notin \text{conv}(X_k)$, since x_k minimizes f over $x \in \text{conv}(X_k)$, so that

  $$\nabla f(x_k)'(x - x_k) \ge 0, \qquad \forall\, x \in \text{conv}(X_k)$$

- Case (b) cannot occur an infinite number of times ($\tilde{x}_{k+1} \notin X_k$ and X has finitely many extreme points), so case (a) must eventually occur.

- The method will find a minimizer of f over X in a finite number of iterations.

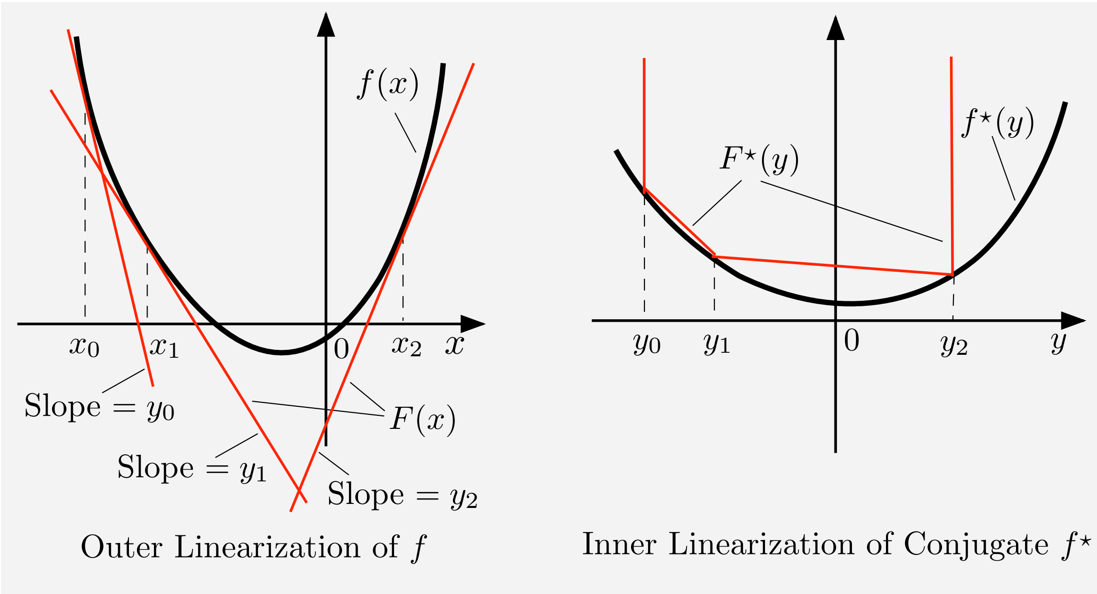
COMMENTS ON SIMPLICIAL DECOMP.

- Important specialized applications
- Variant to enhance efficiency. Discard some of the extreme points that seem unlikely to "participate" in the optimal solution, i.e., all $\tilde{x}$ such that

$$\nabla f(x_{k+1})'(\tilde{x} - x_{k+1}) > 0$$

- Variant to remove the boundedness assumption on X (impose artificial constraints)
- Extension to X nonpolyhedral (method remains unchanged, but convergence proof is more complex)
- Extension to f nondifferentiable (requires use of subgradients in place of gradients, and more sophistication)
- **Duality relation with cutting plane methods**
- We will view cutting plane and simplicial decomposition as special cases of two polyhedral approximation methods that are dual to each other

OUTER LINEARIZATION OF FNS



- Outer linearization of closed proper convex function $f : \Re^n \mapsto (-\infty, \infty]$
- Defined by set of "slopes" $\{y_1, \ldots, y_\ell\}$, where $y_j \in \partial f(x_j)$ for some x_j
- Given by

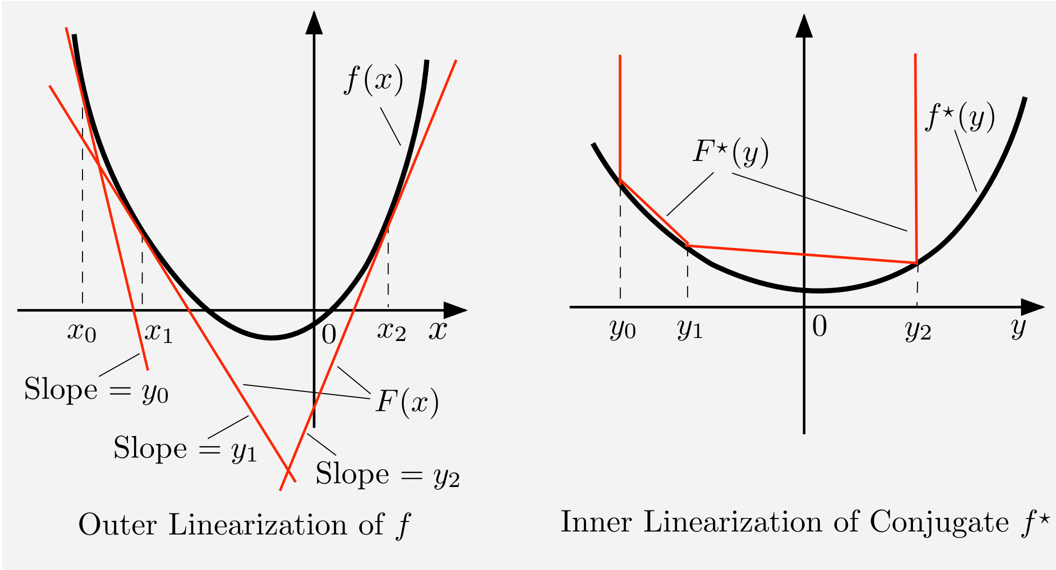
$$F(x) = \max_{j=1,\ldots,\ell} \big\{ f(x_j) + (x - x_j)' y_j \big\}, \qquad x \in \Re^n$$

or equivalently

$$F(x) = \max_{j=1,\ldots,\ell} \big\{ y_j' x - f^\star(y_j) \big\}$$

[this follows using $x_j' y_j = f(x_j) + f^\star(y_j)$, which is implied by $y_j \in \partial f(x_j)$ – the Conjugate Subgradient Theorem]

INNER LINEARIZATION OF FNS



- Consider conjugate $F^\star$ of outer linearization F
- After calculation using the formula

$$F(x) = \max_{j=1,\ldots,\ell} \left\{ y_j' x - f^\star(y_j) \right\}$$

$F^\star$ is a piecewise linear approximation of $f^\star$ defined by “break points” at $y_1, \ldots, y_\ell$

- We have

$$\mathrm{dom}(F^\star) = \mathrm{conv}\big(\{y_1, \ldots, y_\ell\}\big),$$

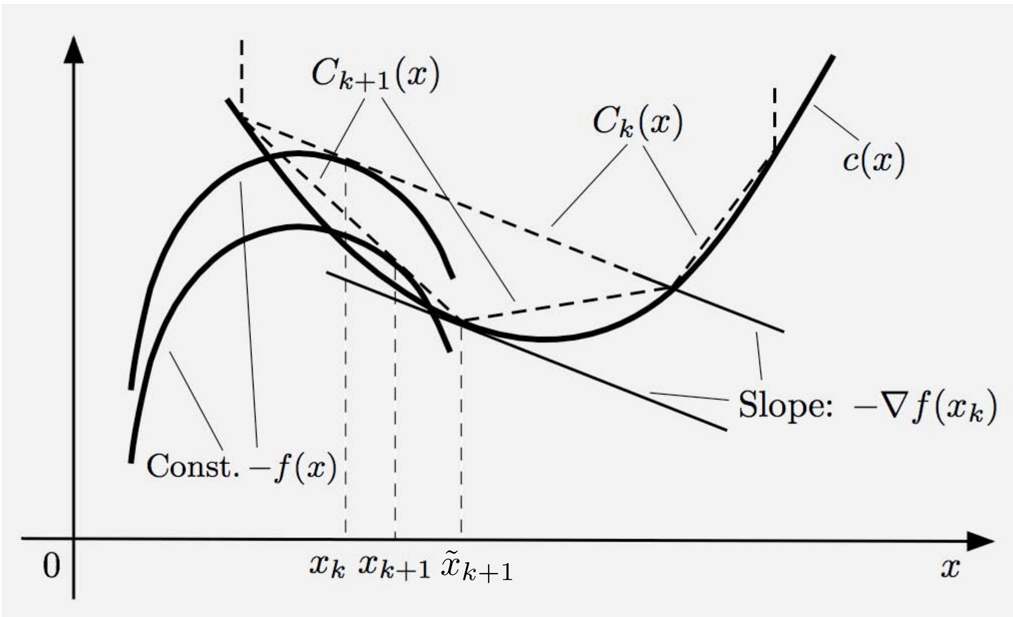
with values at $y_1, \ldots, y_\ell$ equal to the corresponding values of $f^\star$

- Epigraph of $F^\star$ is the convex hull of the union of the vertical halflines corresponding to $y_1, \ldots, y_\ell$:

$$\mathrm{epi}(F^\star) = \mathrm{conv}\Big(\cup_{j=1,\ldots,\ell} \big\{ (y_j, w) \mid f^\star(y_j) \le w \big\} \Big)$$

GENERALIZED SIMPLICIAL DECOMPOSITION

- Consider minimization of $f(x) + c(x)$, over $x \in \Re^n$, where f and c are closed proper convex
- Case where f is differentiable



- Given C_k: inner linearization of c, obtain

$$x_k \in \arg \min_{x \in \Re^n} \big\{ f(x) + C_k(x) \big\}$$

- Obtain $\tilde{x}_{k+1}$ such that

$$-\nabla f(x_k) \in \partial c(\tilde{x}_{k+1}),$$

and form $X_{k+1} = X_k \cup \{\tilde{x}_{k+1}\}$

NONDIFFERENTIABLE CASE

- Given C_k: inner linearization of c, obtain

$$x_k \in \arg\min_{x \in \Re^n} \{ f(x) + C_k(x) \}$$

- Obtain a subgradient $g_k \in \partial f(x_k)$ such that

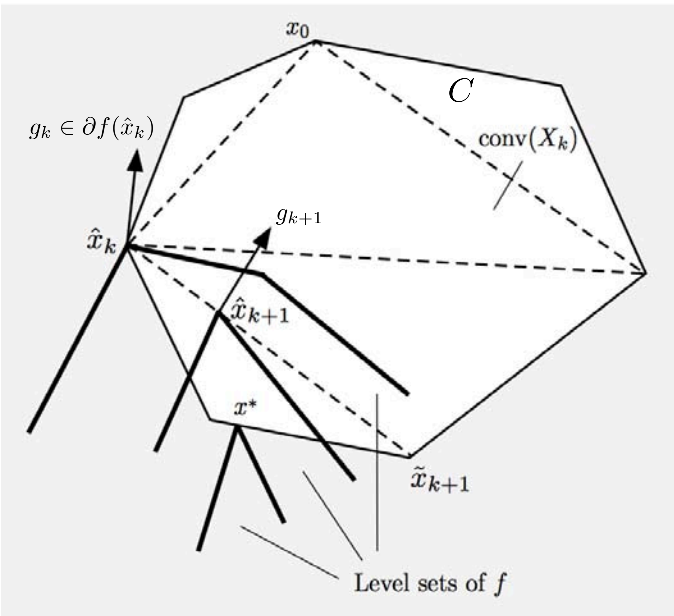
$$-g_k \in \partial C_k(x_k)$$

- Obtain $\tilde{x}_{k+1}$ such that

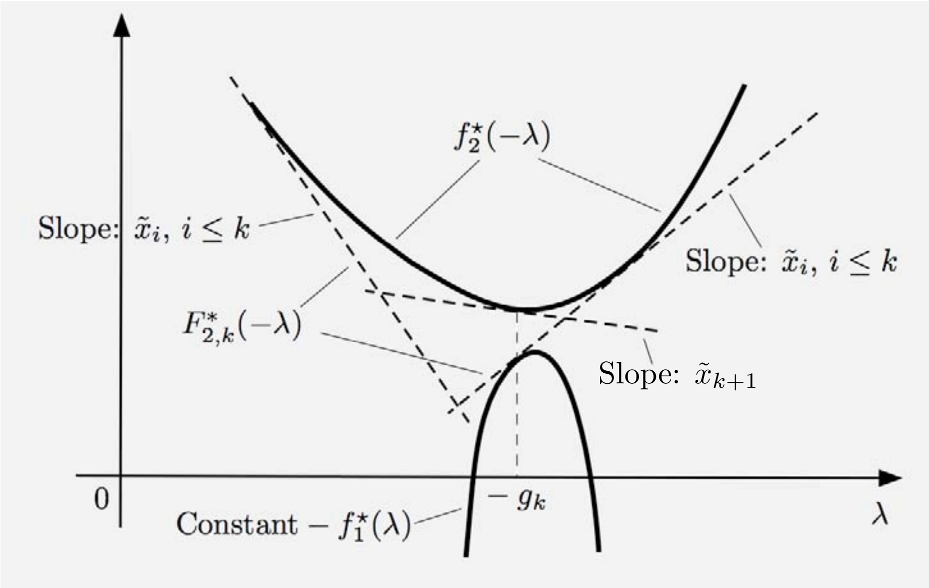
$$-g_k \in \partial c(\tilde{x}_{k+1}),$$

and form $X_{k+1} = X_k \cup \{\tilde{x}_{k+1}\}$

- **Example:** c is the indicator function of a polyhedral set



DUAL CUTTING PLANE IMPLEMENTATION



- Primal and dual Fenchel pair

$$\min_{x\in\Re^n} f_1(x) + f_2(x), \qquad \min_{\lambda\in\Re^n} f_1^\star(\lambda) + f_2^\star(-\lambda)$$

- Primal and dual approximations

$$\min_{x\in\Re^n} f_1(x) + F_{2,k}(x) \qquad \min_{\lambda\in\Re^n} f_1^\star(\lambda) + F_{2,k}^\star(-\lambda)$$

- $F_{2,k}$ and $F_{2,k}^\star$ are inner and outer approximations of f_2 and $f_2^\star$

- $\tilde{x}_{i+1}$ and g_i are solutions of the primal or the dual approximating problem (and corresponding subgradients)