LECTURE 16

LECTURE OUTLINE

- Approximate subgradient methods
- Approximation methods
- Cutting plane methods

APPROXIMATE SUBGRADIENT METHODS

- Consider minimization of

$$f(x) = \sup_{z \in Z} \phi(x, z)$$

where $Z \subset \Re^m$ and $\phi(\cdot, z)$ is convex for all $z \in Z$ (dual minimization is a special case).

- To compute subgradients of f at $x \in \text{dom}(f)$, we find $z_x \in Z$ attaining the supremum above. Then

$$g_x \in \partial \phi(x, z_x) \qquad \Rightarrow \qquad g_x \in \partial f(x)$$

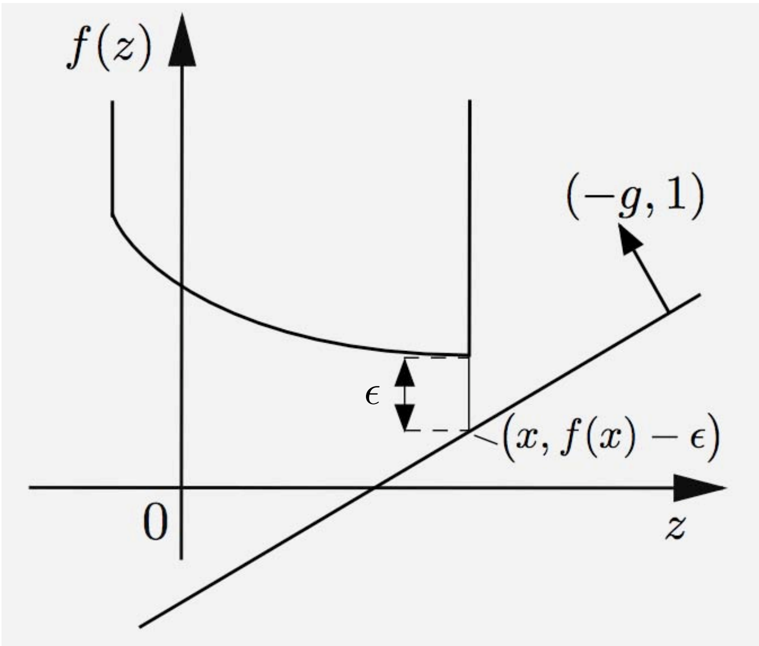
- **Potential difficulty:** For subgradient method, we need to solve exactly the above maximization over $z \in Z$.

- We consider methods that use "approximate" subgradients that can be computed more easily.

ϵ-SUBDIFFERENTIAL

- Fot a proper convex $f : \Re^n \mapsto (-\infty, \infty]$ and $\epsilon > 0$, we say that a vector g is an *ϵ-subgradient* of f at a point $x \in \text{dom}(f)$ if

$$f(z) \geq f(x) + (z - x)'g - \epsilon, \qquad \forall\ z \in \Re^n$$



- The *ϵ-subdifferential* $\partial_\epsilon f(x)$ is the set of all ϵ-subgradients of f at x. By convention, $\partial_\epsilon f(x) = \emptyset$ for $x \notin \text{dom}(f)$.

- We have $\cap_{\epsilon \downarrow 0} \partial_\epsilon f(x) = \partial f(x)$ and

$$\partial_{\epsilon_1} f(x) \subset \partial_{\epsilon_2} f(x) \quad \text{ if } 0 < \epsilon_1 < \epsilon_2$$

CALCULATION OF AN ϵ-SUBGRADIENT

- Consider minimization of

$$f(x) = \sup_{z \in Z} \phi(x, z), \tag{1}$$

where $x \in \Re^n$, $z \in \Re^m$, Z is a subset of $\Re^m$, and $\phi : \Re^n \times \Re^m \mapsto (-\infty, \infty]$ is a function such that $\phi(\cdot, z)$ is convex and closed for each $z \in Z$.

- How to calculate ϵ-subgradient at $x \in \mathrm{dom}(f)$?
- Let $z_x \in Z$ attain the supremum within $\epsilon \geq 0$ in Eq. (1), and let g_x be some subgradient of the convex function $\phi(\cdot, z_x)$.
- For all $y \in \Re^n$, using the subgradient inequality,

$$\begin{aligned} f(y) = \sup_{z \in Z} \phi(y, z) &\geq \phi(y, z_x) \\ &\geq \phi(x, z_x) + g_x'(y - x) \geq f(x) - \epsilon + g_x'(y - x) \end{aligned}$$

i.e., g_x is an ϵ-subgradient of f at x, so

$$\phi(x, z_x) \geq \sup_{z \in Z} \phi(x, z) - \epsilon \text{ and } g_x \in \partial\phi(x, z_x)$$

$$\Rightarrow \qquad g_x \in \partial_\epsilon f(x)$$

ϵ-SUBGRADIENT METHOD

- Uses an ϵ-subgradient in place of a subgradient.

- **Problem:** Minimize convex $f : \Re^n \mapsto \Re$ over a closed convex set X.

- **Method:**

$$x_{k+1} = P_X(x_k - \alpha_k g_k)$$

where g_k is an ϵ_k-subgradient of f at x_k, α_k is a positive stepsize, and $P_X(\cdot)$ denotes projection on X.

- Can be viewed as subgradient method with "errors".

CONVERGENCE ANALYSIS

• **Basic inequality:** If $\{x_k\}$ is the ϵ-subgradient method sequence, for all $y \in X$ and $k \geq 0$

$$\|x_{k+1}-y\|^2 \leq \|x_k-y\|^2 - 2\alpha_k\big(f(x_k)-f(y)-\epsilon_k\big)+\alpha_k^2\|g_k\|^2$$

• Replicate the entire convergence analysis for subgradient methods, but carry along the ϵ_k terms.

• **Example:** Constant $\alpha_k \equiv \alpha$, constant $\epsilon_k \equiv \epsilon$. Assume $\|g_k\| \leq c$ for all k. For any optimal x^*,

$$\|x_{k+1}-x^*\|^2 \leq \|x_k-x^*\|^2 - 2\alpha\big(f(x_k)-f^*-\epsilon\big)+\alpha^2c^2,$$

so the distance to x^* decreases if

$$0 < \alpha < \frac{2\big(f(x_k) - f^* - \epsilon\big)}{c^2}$$

or equivalently, if x_k is outside the level set

$$\left\{x \;\middle|\; f(x) \leq f^* + \epsilon + \frac{\alpha c^2}{2}\right\}$$

• **Example:** If $\alpha_k \to 0$, $\sum_k \alpha_k \to \infty$, and $\epsilon_k \to \epsilon$, we get convergence to the ϵ-optimal set.

INCREMENTAL SUBGRADIENT METHODS

- Consider minimization of sum

$$f(x) = \sum_{i=1}^{m} f_i(x)$$

- Often arises in duality contexts with m: **very large** (e.g., separable problems).

- Incremental method **moves x along a subgradient g_i of a component function f_i** NOT the (expensive) subgradient of f, which is $\sum_i g_i$.

- View an iteration as a cycle of m subiterations, one for each component f_i.

- Let x_k be obtained after k cycles. To obtain x_{k+1}, do one more cycle: Start with $\psi_0 = x_k$, and set $x_{k+1} = \psi_m$, after the m steps

$$\psi_i = P_X(\psi_{i-1} - \alpha_k g_i), \qquad i = 1, \ldots, m$$

with g_i being a subgradient of f_i at ψ_{i-1}.

- **Motivation is faster convergence.** A cycle can make much more progress than a subgradient iteration with essentially the same computation.

CONNECTION WITH ϵ-SUBGRADIENTS

- **Neighborhood property:** If x and $\overline{x}$ are "near" each other, then subgradients at $\overline{x}$ can be viewed as ϵ-subgradients at x, with ϵ "small."

- If $g \in \partial f(\overline{x})$, we have for all $z \in \Re^n$,

$$\begin{aligned} f(z) &\geq f(\overline{x}) + g'(z - \overline{x}) \\ &\geq f(x) + g'(z - x) + f(\overline{x}) - f(x) + g'(x - \overline{x}) \\ &\geq f(x) + g'(z - x) - \epsilon, \end{aligned}$$

where $\epsilon = |f(\overline{x}) - f(x)| + \|g\| \cdot \|\overline{x} - x\|$. Thus, $g \in \partial_\epsilon f(x)$, with ϵ: small when $\overline{x}$ is near x.

- The incremental subgradient iter. is an ϵ-subgradient iter. with $\epsilon = \epsilon_1 + \cdots + \epsilon_m$, where ϵ_i is the "error" in ith step in the cycle (ϵ_i: Proportional to α_k).

- Use

$$\partial_{\epsilon_1} f_1(x) + \cdots + \partial_{\epsilon_m} f_m(x) \subset \partial_\epsilon f(x),$$

where $\epsilon = \epsilon_1 + \cdots + \epsilon_m$, to approximate the ϵ-subdifferential of the sum $f = \sum_{i=1}^m f_i$.

- Convergence to optimal if $\alpha_k \to 0$, $\sum_k \alpha_k \to \infty$.

APPROXIMATION APPROACHES

- Approximation methods replace the original problem with an approximate problem.
- The approximation may be iteratively refined, for convergence to an exact optimum.
- A partial list of methods:
  - Cutting plane/outer approximation.
  - Simplicial decomposition/inner approximation.
  - Proximal methods (including Augmented Lagrangian methods for constrained minimization).
  - Interior point methods.
- A partial list of combination of methods:
  - Combined inner-outer approximation.
  - Bundle methods (proximal-cutting plane).
  - Combined proximal-subgradient (incremental option).

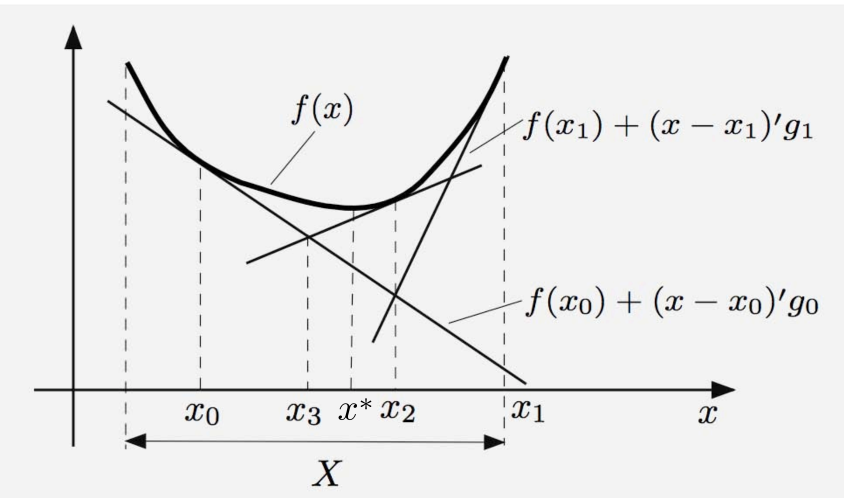
SUBGRADIENTS-OUTER APPROXIMATION

- Consider minimization of a convex function $f : \Re^n \mapsto \Re$, over a closed convex set X.
- We assume that at each $x \in X$, a subgradient g of f can be computed.
- We have

$$f(z) \geq f(x) + g'(z - x), \qquad \forall\, z \in \Re^n,$$

so each subgradient defines a plane (a linear function) that approximates f from below.

- The idea of the outer approximation/cutting plane approach is to build an ever more accurate approximation of f using such planes.



CUTTING PLANE METHOD

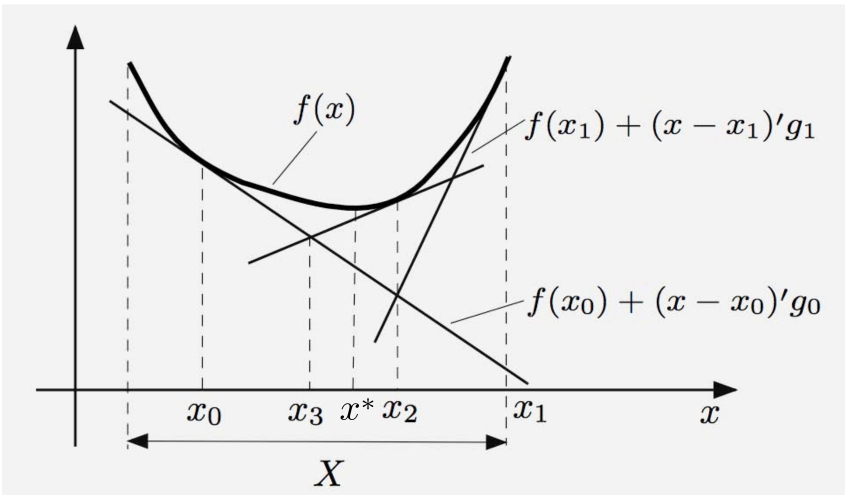
- Start with any $x_0 \in X$. For $k \geq 0$, set

$$x_{k+1} \in \arg\min_{x \in X} F_k(x),$$

where

$$F_k(x) = \max\Big\{f(x_0)+(x-x_0)'g_0, \ldots, f(x_k)+(x-x_k)'g_k\Big\}$$

and g_i is a subgradient of f at x_i.



- Note that $F_k(x) \leq f(x)$ for all x, and that $F_k(x_{k+1})$ increases monotonically with k. These imply that all limit points of x_k are optimal.

Proof: If $x_k \to \overline{x}$ then $F_k(x_k) \to f(\overline{x})$, [otherwise there would exist a hyperplane strictly separating $\text{epi}(f)$ and $(\overline{x}, \lim_{k\to\infty} F_k(x_k))$]. This implies that $f(\overline{x}) \leq \lim_{k\to\infty} F_k(x) \leq f(x)$ for all x. **Q.E.D.**

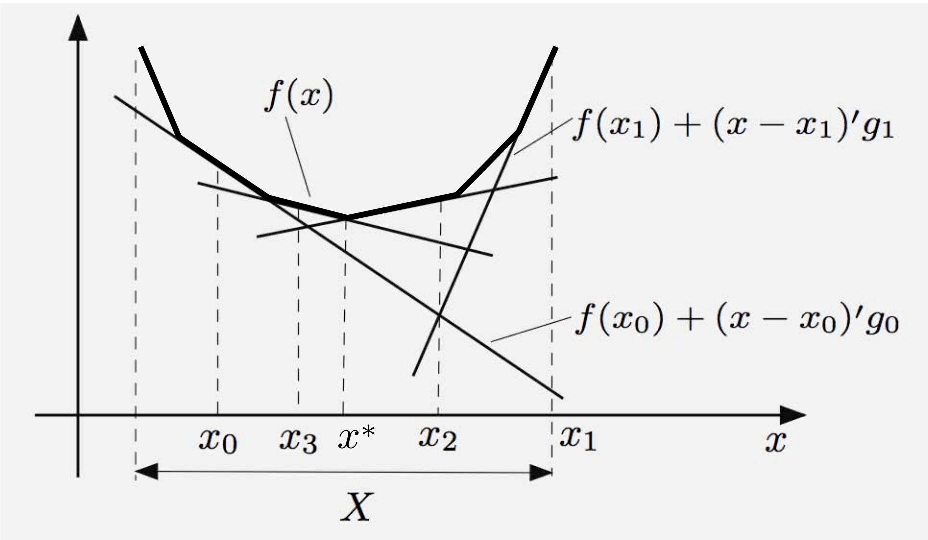
CONVERGENCE AND TERMINATION

- We have for all k

$$F_k(x_{k+1}) \leq f^* \leq \min_{i \leq k} f(x_i)$$

- Termination when $\min_{i \leq k} f(x_i) - F_k(x_{k+1})$ comes to within some small tolerance.

- For f polyhedral, we have finite termination with an exactly optimal solution.



- **Instability problem:** The method can make large moves that deteriorate the value of f.

- Starting from the exact minimum it typically moves away from that minimum.

VARIANTS

- **Variant I:** Simultaneously with f, construct polyhedral approximations to X.
- **Variant II:** Central cutting plane methods