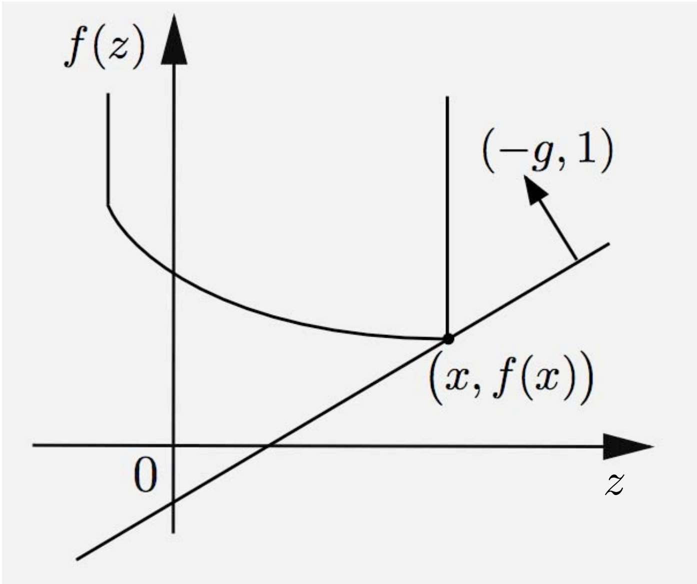
LECTURE 12

LECTURE OUTLINE

- Subgradients
- Fenchel inequality
- Sensitivity in constrained optimization
- Subdifferential calculus
- Optimality conditions

Reading: Section 5.4

SUBGRADIENTS



- Let $f : \Re^n \mapsto (-\infty, \infty]$ be a convex function. A vector $g \in \Re^n$ is a *subgradient* of f at a point $x \in \text{dom}(f)$ if

$$f(z) \geq f(x) + (z - x)'g, \qquad \forall \; z \in \Re^n$$

- **Support Hyperplane Interpretation:** g is a subgradient if and only if

$$f(z) - z'g \geq f(x) - x'g, \qquad \forall \; z \in \Re^n$$

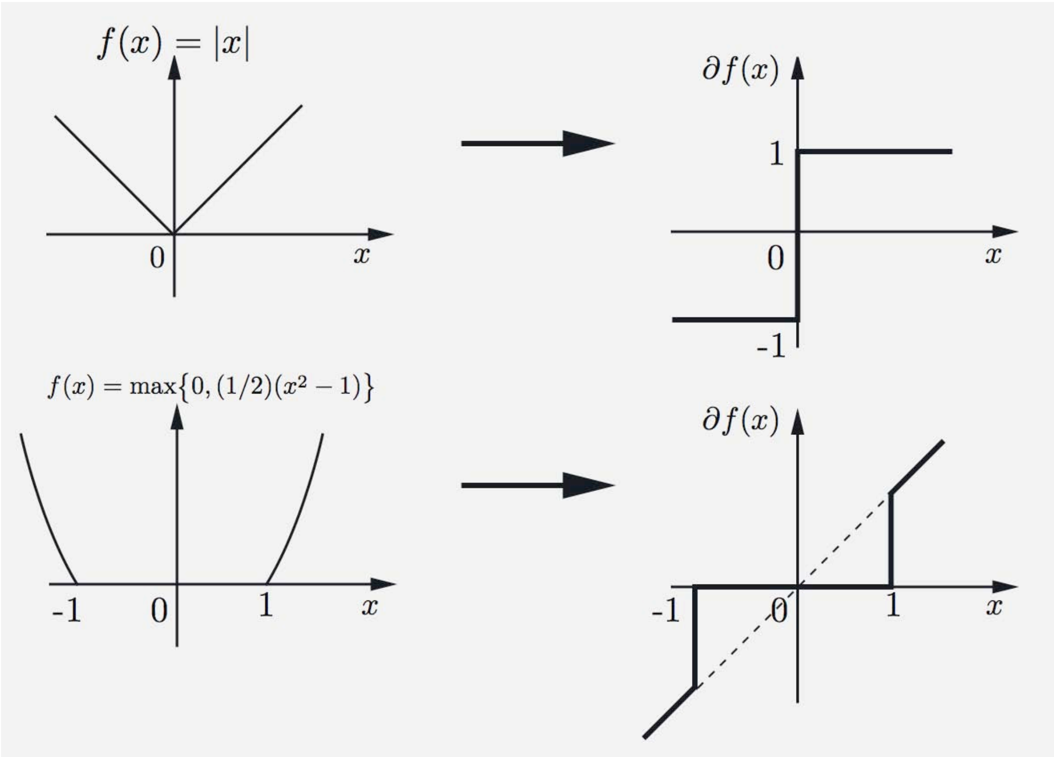
so g is a subgradient at x if and only if the hyperplane in $\Re^{n+1}$ that has normal $(-g, 1)$ and passes through $\big(x, f(x)\big)$ supports the epigraph of f.

- The set of all subgradients at x is the *subdifferential of f at x*, denoted $\partial f(x)$.

- By convention $\partial f(x) = \emptyset$ for $x \notin \text{dom}(f)$.

EXAMPLES OF SUBDIFFERENTIALS

- Some examples:



- If f is differentiable, then $\partial f(x) = \{\nabla f(x)\}$.

Proof: If $g \in \partial f(x)$, then

$$f(x+z) \geq f(x) + g'z, \qquad \forall\ z \in \Re^n.$$

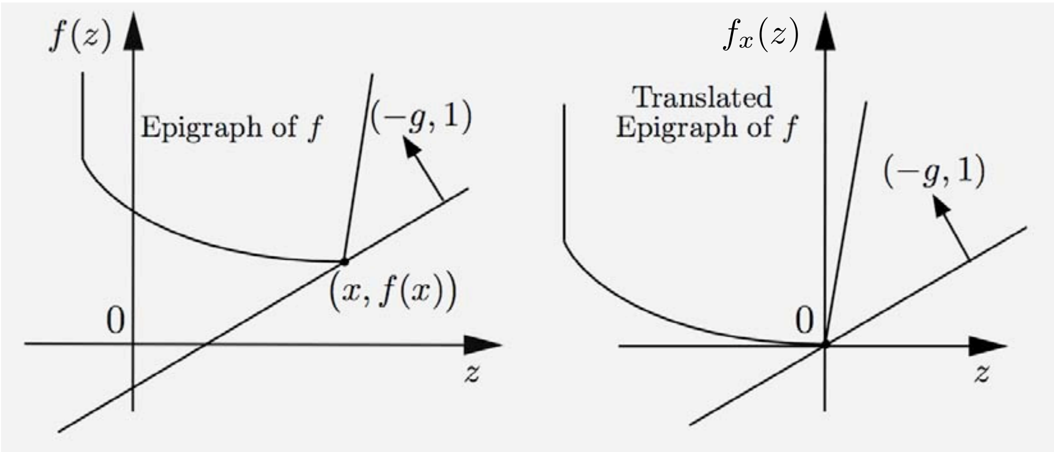
Apply this with $z = \gamma\big(\nabla f(x) - g\big)$, $\gamma \in \Re$, and use 1st order Taylor series expansion to obtain

$$\|\nabla f(x) - g\|^2 \leq -o(\gamma)/\gamma, \qquad \forall\ \gamma < 0$$

EXISTENCE OF SUBGRADIENTS

- Let $f : \Re^n \mapsto (-\infty, \infty]$ be proper convex.
- Consider MC/MC with

$$M = \text{epi}(f_x), \qquad f_x(z) = f(x+z) - f(x)$$



- By 2nd MC/MC Duality Theorem, $\partial f(x)$ is nonempty and compact if and only if x is in the interior of dom(f).
- More generally: for every $x \in \text{ri}\big(\text{dom}(f)\big)$,

$$\partial f(x) = S^\perp + G,$$

where:

– S is the subspace that is parallel to the affine hull of dom(f)

– G is a nonempty and compact set.

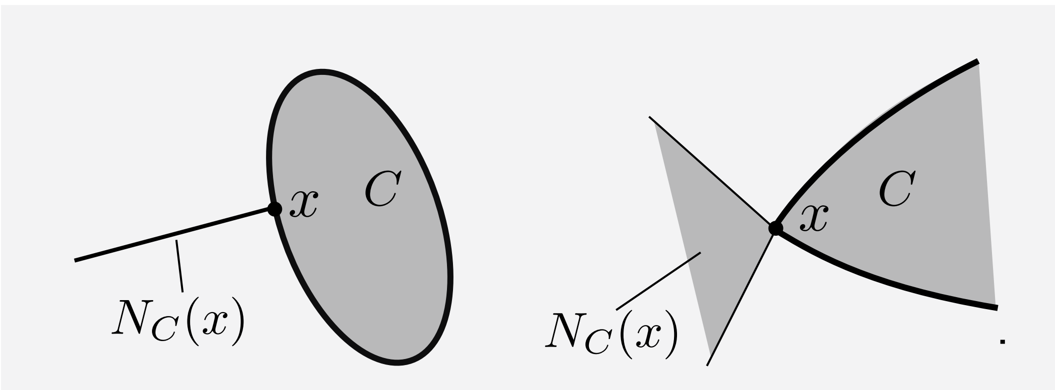
EXAMPLE: SUBDIFFERENTIAL OF INDICATOR

- Let C be a convex set, and δ_C be its indicator function.
- For $x \notin C$, $\partial\delta_C(x) = \emptyset$ (by convention).
- For $x \in C$, we have $g \in \partial\delta_C(x)$ iff

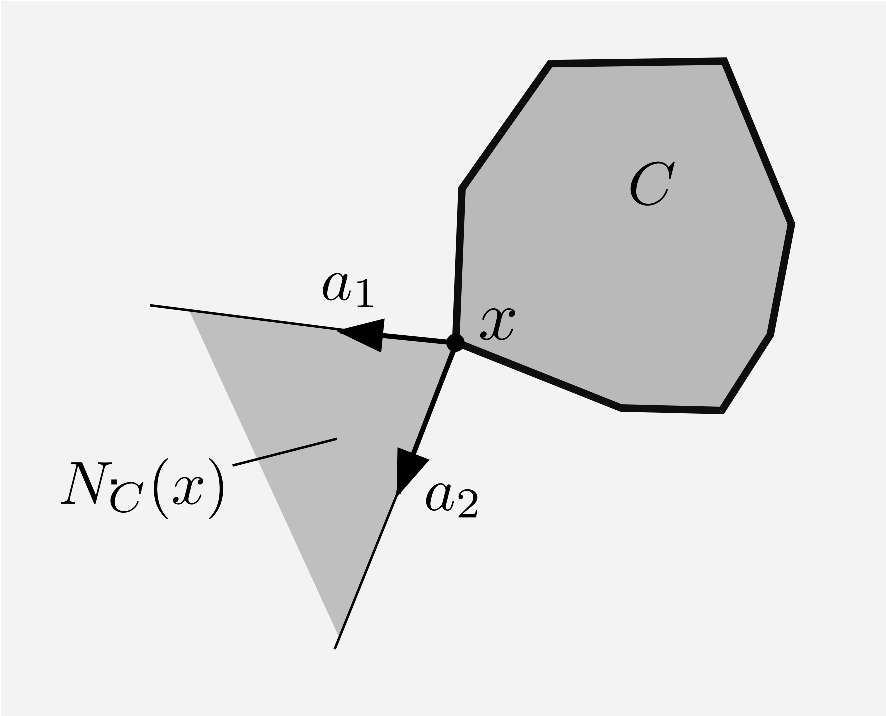
$$\delta_C(z) \geq \delta_C(x) + g'(z - x), \qquad \forall\ z \in C,$$

or equivalently $g'(z - x) \leq 0$ for all $z \in C$. Thus $\partial\delta_C(x)$ is the *normal cone of* C *at* x, denoted $N_C(x)$:

$$N_C(x) = \big\{ g \mid g'(z - x) \leq 0,\ \forall\ z \in C \big\}.$$



EXAMPLE: POLYHEDRAL CASE



- For the case of a polyhedral set

$$C = \{x \mid a_i' x \le b_i,\ i = 1, \dots, m\},$$

we have

$$N_C(x) = \begin{cases} \{0\} & \text{if } x \in \text{int}(C), \\ \text{cone}\big(\{a_i \mid a_i' x = b_i\}\big) & \text{if } x \notin \text{int}(C). \end{cases}$$

- **Proof:** Given x, disregard inequalities with $a_i' x < b_i$, and translate C to move x to 0, so it becomes a cone. The polar cone is $N_C(x)$.

FENCHEL INEQUALITY

• Let $f : \Re^n \mapsto (-\infty, \infty]$ be proper convex and let $f^\star$ be its conjugate. Using the definition of conjugacy, we have *Fenchel's inequality*:

$$x'y \leq f(x) + f^\star(y), \qquad \forall\, x \in \Re^n,\ y \in \Re^n.$$

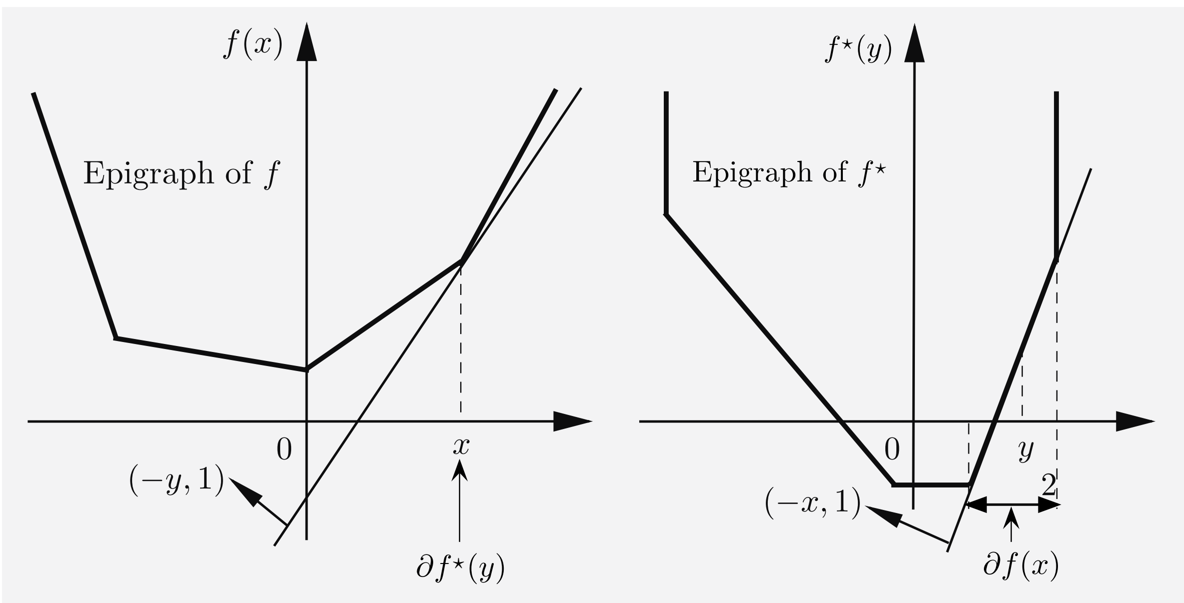
• **Conjugate Subgradient Theorem:** The following two relations are equivalent for a pair of vectors (x, y):

(i) $x'y = f(x) + f^\star(y)$.

(ii) $y \in \partial f(x)$.

If f is closed, (i) and (ii) are equivalent to

(iii) $x \in \partial f^\star(y)$.



MINIMA OF CONVEX FUNCTIONS

• **Application:** Let f be closed proper convex and let X^* be the set of minima of f over $\Re^n$. Then:

(a) $X^* = \partial f^\star(0)$.

(b) X^* is nonempty if $0 \in \mathrm{ri}\big(\mathrm{dom}(f^\star)\big)$.

(c) X^* is nonempty and compact if and only if $0 \in \mathrm{int}\big(\mathrm{dom}(f^\star)\big)$.

Proof: (a) We have $x^* \in X^*$ iff $f(x) \geq f(x^*)$ for all $x \in \Re^n$. So

$$x^* \in X^* \qquad \text{iff} \qquad 0 \in \partial f(x^*) \qquad \text{iff} \qquad x^* \in \partial f^\star(0)$$

where:

– 1st relation follows from the subgradient inequality

– 2nd relation follows from the conjugate subgradient theorem

(b) $\partial f^\star(0)$ is nonempty if $0 \in \mathrm{ri}\big(\mathrm{dom}(f^\star)\big)$.

(c) $\partial f^\star(0)$ is nonempty and compact if and only if $0 \in \mathrm{int}\big(\mathrm{dom}(f^\star)\big)$. **Q.E.D.**

SENSITIVITY INTERPRETATION

- Consider MC/MC for the case $M = \text{epi}(p)$.
- Dual function is

$$q(\mu) = \inf_{u \in \Re^m} \{p(u) + \mu' u\} = -p^\star(-\mu),$$

where $p^\star$ is the conjugate of p.

- Assume p is proper convex and strong duality holds, so $p(0) = w^* = q^* = \sup_{\mu \in \Re^m} \{-p^\star(-\mu)\}$. Let Q^* be the set of dual optimal solutions,

$$Q^* = \{\mu^* \mid p(0) + p^\star(-\mu^*) = 0\}.$$

From Conjugate Subgradient Theorem, $\mu^* \in Q^*$ if and only if $-\mu^* \in \partial p(0)$, i.e., $Q^* = -\partial p(0)$.

- If p is convex and differentiable at 0, $-\nabla p(0)$ is equal to the unique dual optimal solution μ^*.
- **Constrained optimization example:**

$$p(u) = \inf_{x \in X,\, g(x) \leq u} f(x),$$

If p is convex and differentiable,

$$\mu_j^* = -\frac{\partial p(0)}{\partial u_j}, \qquad j = 1, \ldots, r.$$

EXAMPLE: SUBDIFF. OF SUPPORT FUNCTION

- Consider the support function $\sigma_X(y)$ of a set X. To calculate $\partial\sigma_X(\overline{y})$ at some $\overline{y}$, we introduce

$$r(y) = \sigma_X(y + \overline{y}), \qquad y \in \Re^n.$$

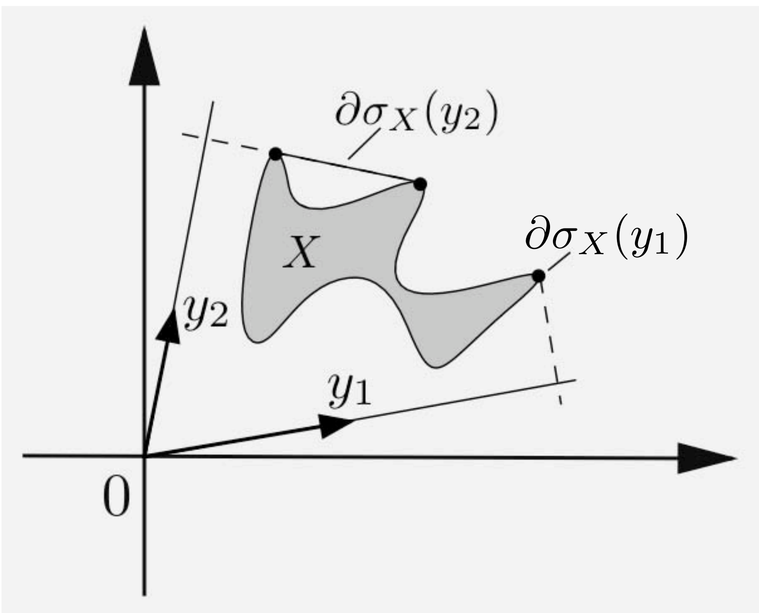
- We have $\partial\sigma_X(\overline{y}) = \partial r(0) = \arg\min_{x\in\Re^n} r^\star(x)$.
- We have $r^\star(x) = \sup_{y\in\Re^n}\{y'x - r(y)\}$, or

$$r^\star(x) = \sup_{y\in\Re^n}\{y'x - \sigma_X(y + \overline{y})\} = \delta(x) - \overline{y}'x,$$

where δ is the indicator function of $\text{cl}\big(\text{conv}(X)\big)$.

- Hence $\partial\sigma_X(\overline{y}) = \arg\min_{x\in\Re^n}\big\{\delta(x) - \overline{y}'x\big\}$, or

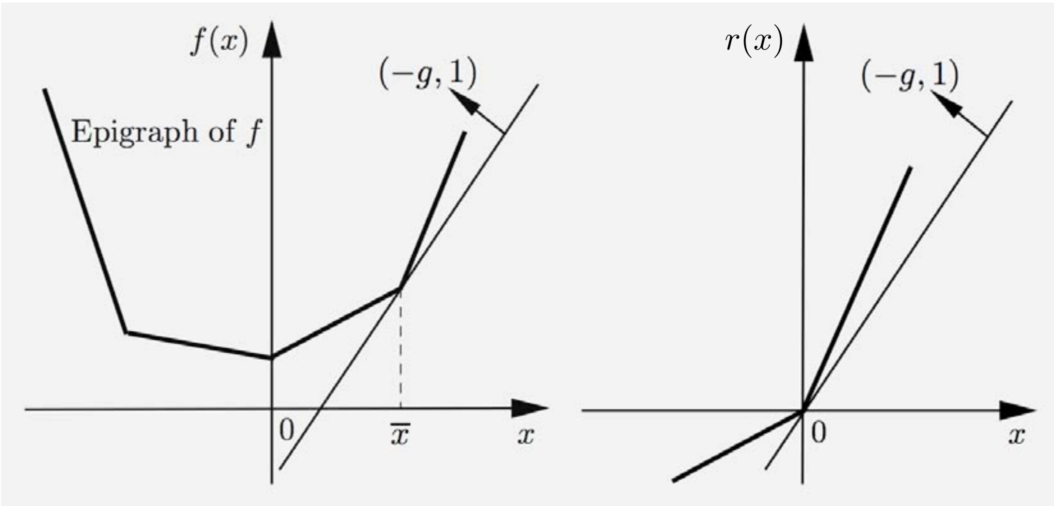
$$\partial\sigma_X(\overline{y}) = \arg \max_{x\in \text{cl}\left(\text{conv}(X)\right)} \overline{y}'x$$



EXAMPLE: SUBDIFF. OF POLYHEDRAL FN

- Let

$$f(x) = \max\{a_1'x + b_1, \ldots, a_r'x + b_r\}.$$



- For a fixed $\overline{x} \in \Re^n$, consider

$$A_{\overline{x}} = \big\{ j \mid a_j'\overline{x} + b_j = f(\overline{x}) \big\}$$

and the function $r(x) = \max\big\{a_j'x \mid j \in A_{\overline{x}}\big\}$.

- It can be seen that $\partial f(\overline{x}) = \partial r(0)$.

- Since r is the support function of the finite set $\{a_j \mid j \in A_{\overline{x}}\}$, we see that

$$\partial f(\overline{x}) = \partial r(0) = \mathrm{conv}\big(\{a_j \mid j \in A_{\overline{x}}\}\big)$$

CHAIN RULE

- Let $f : \Re^m \mapsto (-\infty, \infty]$ be convex, and A be a matrix. Consider $F(x) = f(Ax)$ and assume that F is proper. If either f is polyhedral or else $\text{Range}(A) \cap \text{ri}(\text{dom}(f)) \neq \emptyset$, then

$$\partial F(x) = A' \partial f(Ax), \qquad \forall\, x \in \Re^n.$$

Proof: Showing $\partial F(x) \supset A' \partial f(Ax)$ is simple and does not require the relative interior assumption. For the reverse inclusion, let $d \in \partial F(x)$ so $F(z) \geq F(x) + (z - x)'d \geq 0$ or $f(Az) - z'd \geq f(Ax) - x'd$ for all z, so (Ax, x) solves

$$\begin{aligned} &\text{minimize} \quad f(y) - z'd \\ &\text{subject to} \ \ y \in \text{dom}(f), \quad Az = y. \end{aligned}$$

If $R(A) \cap \text{ri}(\text{dom}(f)) \neq \emptyset$, by strong duality theorem, there is a dual optimal solution λ, such that

$$(Ax, x) \in \arg \min_{y \in \Re^m,\, z \in \Re^n} \big\{ f(y) - z'd + \lambda'(Az - y) \big\}$$

Since the min over z is unconstrained, we have $d = A'\lambda$, so $Ax \in \arg\min_{y \in \Re^m} \big\{ f(y) - \lambda' y \big\}$, or

$$f(y) \geq f(Ax) + \lambda'(y - Ax), \qquad \forall\, y \in \Re^m.$$

Hence $\lambda \in \partial f(Ax)$, so that $d = A'\lambda \in A' \partial f(Ax)$. It follows that $\partial F(x) \subset A' \partial f(Ax)$. In the polyhedral case, $\text{dom}(f)$ is polyhedral. **Q.E.D.**

SUM OF FUNCTIONS

• Let $f_i : \Re^n \mapsto (-\infty, \infty]$, $i = 1, \ldots, m$, be proper convex functions, and let

$$F = f_1 + \cdots + f_m.$$

• Assume that $\cap_{1=1}^m \text{ri}\big(\text{dom}(f_i)\big) \neq \emptyset$.

• Then

$$\partial F(x) = \partial f_1(x) + \cdots + \partial f_m(x), \qquad \forall\, x \in \Re^n.$$

Proof: We can write F in the form $F(x) = f(Ax)$, where A is the matrix defined by $Ax = (x, \ldots, x)$, and $f : \Re^{mn} \mapsto (-\infty, \infty]$ is the function

$$f(x_1, \ldots, x_m) = f_1(x_1) + \cdots + f_m(x_m).$$

Use the proof of the chain rule.

• **Extension:** If for some k, the functions f_i, $i = 1, \ldots, k$, are polyhedral, it is sufficient to assume

$$\Big(\cap_{i=1}^k \text{dom}(f_i) \Big) \cap \Big(\cap_{i=k+1}^m \text{ri}\big(\text{dom}(f_i)\big) \Big) \neq \emptyset.$$

CONSTRAINED OPTIMALITY CONDITION

- Let $f : \Re^n \mapsto (-\infty, \infty]$ be proper convex, let X be a convex subset of $\Re^n$, and assume that one of the following four conditions holds:

  (i) $\mathrm{ri}\big(\mathrm{dom}(f)\big) \cap \mathrm{ri}(X) \neq \emptyset$.

  (ii) f is polyhedral and $\mathrm{dom}(f) \cap \mathrm{ri}(X) \neq \emptyset$.

  (iii) X is polyhedral and $\mathrm{ri}\big(\mathrm{dom}(f)\big) \cap X \neq \emptyset$.

  (iv) f and X are polyhedral, and $\mathrm{dom}(f) \cap X \neq \emptyset$.

  Then, a vector x^* minimizes f over X iff there exists $g \in \partial f(x^*)$ such that $-g$ belongs to the normal cone $N_X(x^*)$, i.e.,

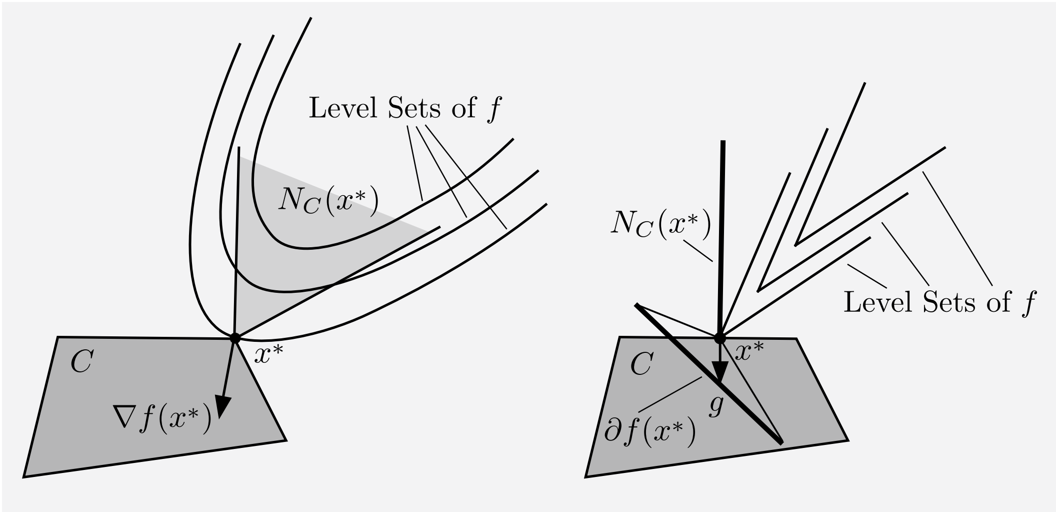
$$g'(x - x^*) \geq 0, \qquad \forall\, x \in X.$$

Proof: x^* minimizes

$$F(x) = f(x) + \delta_X(x)$$

if and only if $0 \in \partial F(x^*)$. Use the formula for subdifferential of sum. **Q.E.D.**

ILLUSTRATION OF OPTIMALITY CONDITION



- In the figure on the left, f is differentiable and the condition is that

$$-\nabla f(x^*) \in N_C(x^*),$$

which is equivalent to

$$\nabla f(x^*)'(x - x^*) \geq 0, \qquad \forall\ x \in X.$$

- In the figure on the right, f is nondifferentiable, and the condition is that

$$-g \in N_C(x^*) \quad \text{for some } g \in \partial f(x^*).$$

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