LECTURE 10

LECTURE OUTLINE

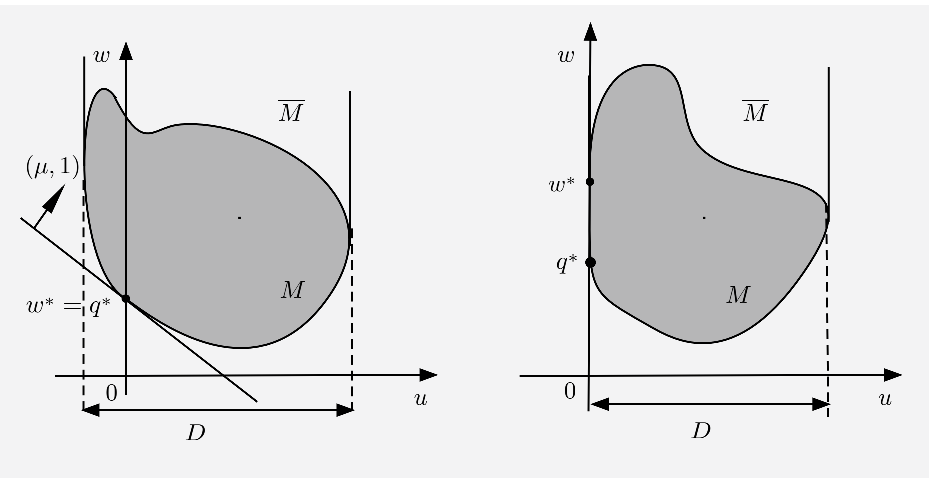
- Min Common/Max Crossing Th. III
- Nonlinear Farkas Lemma/Linear Constraints
- Linear Programming Duality
- Convex Programming Duality
- Optimality Conditions

Reading: Sections 4.5, 5.1,5.2, 5.3.1, 5.3.2

Recall the MC/MC Theorem II: If $-\infty < w^*$ and

$$0 \in \mathrm{ri}(D) = \big\{u \mid \text{there exists } w \in \Re \text{ with } (u, w) \in \overline{M}\big\}$$

then $q^* = w^*$ and there exists μ s. t. $q(\mu) = q^*$.



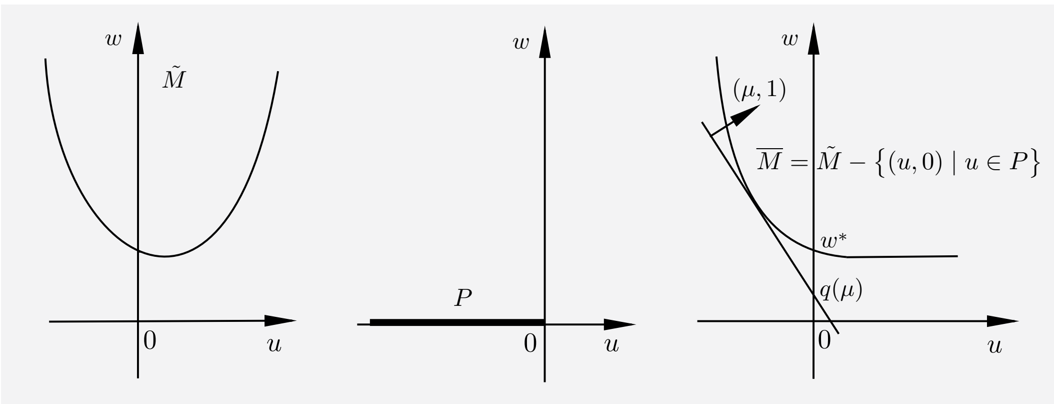
MC/MC TH. III - POLYHEDRAL

- Consider the MC/MC problems, and assume that $-\infty < w^*$ and:

  (1) $\overline{M}$ is a "horizontal translation" of $\tilde{M}$ by $-P$,

$$\overline{M} = \tilde{M} - \big\{(u, 0) \mid u \in P\big\},$$

where P: polyhedral and $\tilde{M}$: convex.



  (2) We have $\mathrm{ri}(\tilde{D}) \cap P \neq \emptyset$, where

$$\tilde{D} = \big\{u \mid \text{there exists } w \in \Re \text{ with } (u, w) \in \tilde{M}\big\}$$

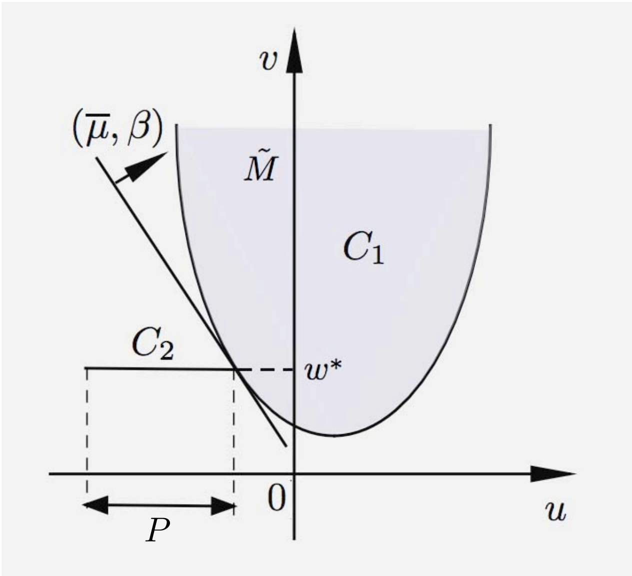
Then $q^* = w^*$, there is a max crossing solution, and all max crossing solutions $\overline{\mu}$ satisfy $\overline{\mu}'d \leq 0$ for all $d \in R_P$.

- **Comparison with Th. II:** Since $D = \tilde{D} - P$, the condition $0 \in \mathrm{ri}(D)$ of Theorem II is

$$\mathrm{ri}(\tilde{D}) \cap \mathrm{ri}(P) \neq \emptyset$$

PROOF OF MC/MC TH. III

- Consider the *disjoint* convex sets $C_1 = \big\{(u,v) \mid v > w \text{ for some } (u,w) \in \tilde{M}\big\}$ and $C_2 = \big\{(u,w^*) \mid u \in P\big\}$ [$u \in P$ and $(u,w) \in \tilde{M}$ with $w^* > w$ contradicts the definition of w^*]



- Since C_2 is polyhedral, there exists a separating hyperplane not containing C_1, i.e., a $(\overline{\mu}, \beta) \neq (0,0)$ such that

$$\beta w^* + \overline{\mu}' z \leq \beta v + \overline{\mu}' x, \quad \forall\ (x,v) \in C_1,\ \forall\ z \in P$$

$$\inf_{(x,v)\in C_1} \big\{\beta v + \overline{\mu}' x\big\} < \sup_{(x,v)\in C_1} \big\{\beta v + \overline{\mu}' x\big\}$$

Since $(0,1)$ is a direction of recession of C_1, we see that $\beta \geq 0$. Because of the relative interior point assumption, $\beta \neq 0$, so we may assume that $\beta = 1$.

PROOF (CONTINUED)

- Hence,

$$w^* + \overline{\mu}'z \le \inf_{(u,v)\in C_1} \{v + \overline{\mu}'u\}, \qquad \forall\ z \in P,$$

so that

$$\begin{aligned} w^* &\le \inf_{(u,v)\in C_1,\, z\in P} \big\{v + \overline{\mu}'(u - z)\big\} \\ &= \inf_{(u,v)\in \tilde{M} - P} \{v + \overline{\mu}'u\} \\ &= \inf_{(u,v)\in \overline{M}} \{v + \overline{\mu}'u\} \\ &= q(\overline{\mu}) \end{aligned}$$

Using $q^* \le w^*$ (weak duality), we have $q(\overline{\mu}) = q^* = w^*$.

Proof that all max crossing solutions $\overline{\mu}$ satisfy $\overline{\mu}'d \le 0$ for all $d \in R_P$: follows from

$$q(\mu) = \inf_{(u,v)\in C_1,\, z\in P} \big\{v + \mu'(u - z)\big\}$$

so that $q(\mu) = -\infty$ if $\mu'd > 0$. **Q.E.D.**

- Geometrical intuition: every $(0, -d)$ with $d \in R_P$, is direction of recession of $\overline{M}$.

MC/MC TH. III - A SPECIAL CASE

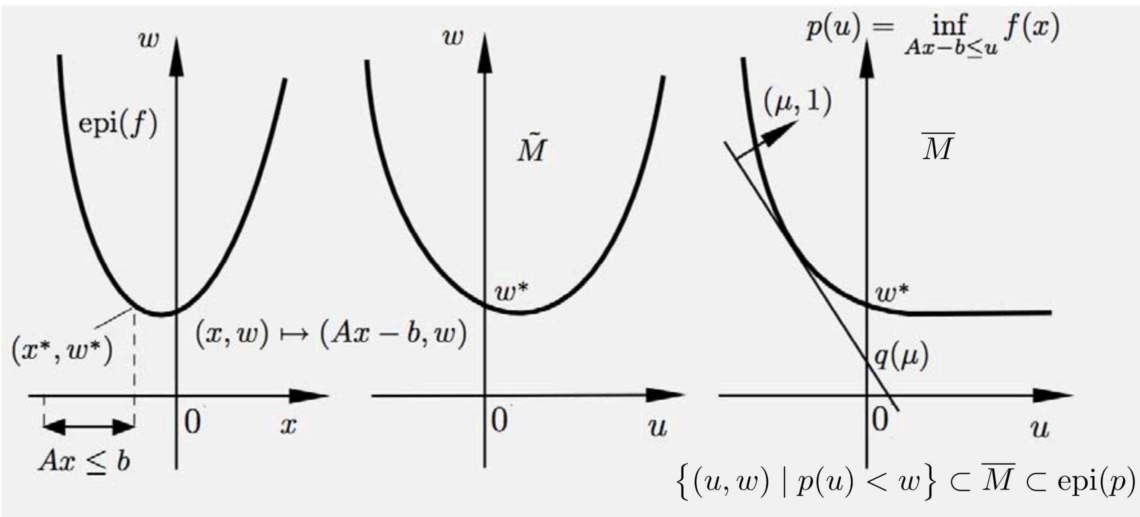
- Consider the MC/MC framework, and assume:

  (1) For a convex function $f : \Re^m \mapsto (-\infty, \infty]$, an $r \times m$ matrix A, and a vector $b \in \Re^r$:

$$\overline{M} = \big\{(u, w) \mid \text{for some } (x, w) \in \text{epi}(f),\ Ax - b \le u\big\}$$

so $\overline{M} = \tilde{M} + $ Positive Orthant, where

$$\tilde{M} = \big\{(Ax - b, w) \mid (x, w) \in \text{epi}(f)\big\}$$



  (2) There is an $\overline{x} \in \text{ri}(\text{dom}(f))$ s. t. $A\overline{x} - b \le 0$.

Then $q^* = w^*$ and there is a $\mu \ge 0$ with $q(\mu) = q^*$.

- Also $\overline{M} = M \approx \text{epi}(p)$, where $p(u) = \inf_{Ax-b\le u} f(x)$.
- We have $w^* = p(0) = \inf_{Ax-b\le 0} f(x)$.

NONL. FARKAS' L. - POLYHEDRAL ASSUM.

- Let $X \subset \Re^n$ be convex, and $f : X \mapsto \Re$ and $g_j : \Re^n \mapsto \Re$, $j = 1, \ldots, r$, be linear so $g(x) = Ax - b$ for some A and b. Assume that

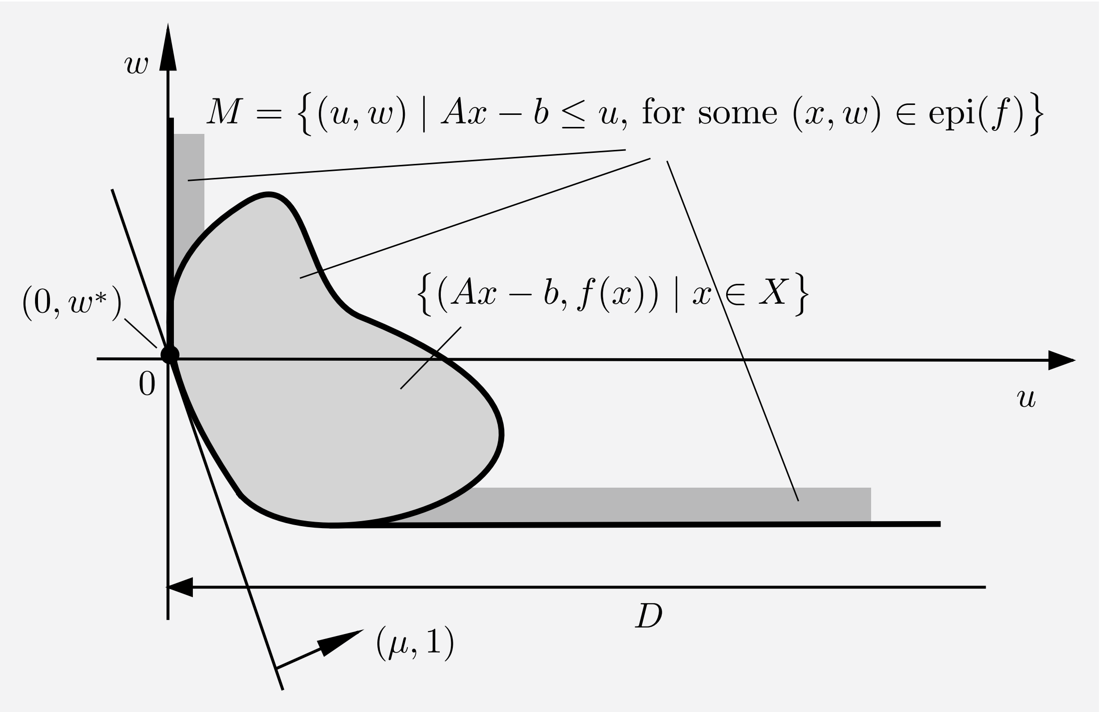
$$f(x) \geq 0, \qquad \forall\, x \in X \text{ with } Ax - b \leq 0$$

Let

$$Q^* = \big\{\mu \mid \mu \geq 0,\ f(x) + \mu'(Ax - b) \geq 0,\ \forall\, x \in X\big\}.$$

Assume that there exists a vector $\overline{x} \in \mathrm{ri}(X)$ such that $A\overline{x} - b \leq 0$. Then Q^* is nonempty.

Proof: As before, apply special case of MC/MC Th. III of preceding slide, using the fact $w^* \geq 0$, implied by the assumption.



(LINEAR) FARKAS' LEMMA

- Let A be an $m \times n$ matrix and $c \in \Re^m$. The system $Ay = c$, $y \geq 0$ has a solution if and only if

$$A'x \leq 0 \qquad \Rightarrow \qquad c'x \leq 0. \qquad (*)$$

- **Alternative/Equivalent Statement:** If $P = \text{cone}\{a_1, \ldots, a_n\}$, where $a_1, \ldots, a_n$ are the columns of A, then $P = (P^*)^*$ (Polar Cone Theorem).

Proof: If $y \in \Re^n$ is such that $Ay = c$, $y \geq 0$, then $y'A'x = c'x$ for all $x \in \Re^m$, which implies Eq. (*).

Conversely, apply the Nonlinear Farkas' Lemma with $f(x) = -c'x$, $g(x) = A'x$, and $X = \Re^m$. Condition (*) implies the existence of $\mu \geq 0$ such that

$$-c'x + \mu'A'x \geq 0, \qquad \forall\, x \in \Re^m,$$

or equivalently

$$(A\mu - c)'x \geq 0, \qquad \forall\, x \in \Re^m,$$

or $A\mu = c$.

LINEAR PROGRAMMING DUALITY

- Consider the linear program

$$\begin{aligned} &\text{minimize} \quad c'x \\ &\text{subject to} \quad a_j'x \geq b_j, \quad j = 1, \ldots, r, \end{aligned}$$

where $c \in \Re^n$, $a_j \in \Re^n$, and $b_j \in \Re$, $j = 1, \ldots, r$.

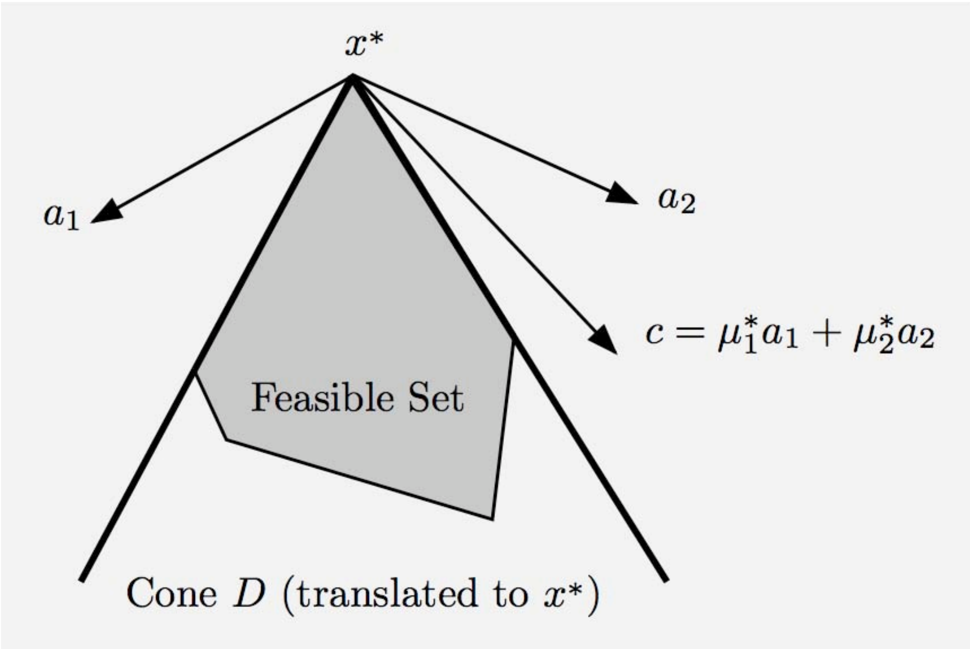
- The dual problem is

$$\begin{aligned} &\text{maximize} \quad b'\mu \\ &\text{subject to} \quad \sum_{j=1}^{r} a_j \mu_j = c, \quad \mu \geq 0. \end{aligned}$$

- **Linear Programming Duality Theorem:**
  (a) If either f^* or q^* is finite, then $f^* = q^*$ and both the primal and the dual problem have optimal solutions.
  (b) If $f^* = -\infty$, then $q^* = -\infty$.
  (c) If $q^* = \infty$, then $f^* = \infty$.

Proof: (b) and (c) follow from weak duality. For part (a): If f^* is finite, there is a primal optimal solution x^*, by existence of solutions of quadratic programs. Use Farkas' Lemma to construct a dual feasible μ^* such that $c'x^* = b'\mu^*$ (next slide).

PROOF OF LP DUALITY (CONTINUED)



- Let x^* be a primal optimal solution, and let $J = \{j \mid a_j' x^* = b_j\}$. Then, $c'y \geq 0$ for all y in the cone of "feasible directions"

$$D = \{y \mid a_j' y \geq 0, \ \forall \ j \in J\}$$

By Farkas' Lemma, for some scalars $\mu_j^* \geq 0$, c can be expressed as

$$c = \sum_{j=1}^{r} \mu_j^* a_j, \quad \mu_j^* \geq 0, \ \forall \ j \in J, \ \ \mu_j^* = 0, \ \forall \ j \notin J.$$

Taking inner product with x^*, we obtain $c'x^* = b'\mu^*$, which in view of $q^* \leq f^*$, shows that $q^* = f^*$ and that μ^* is optimal.

LINEAR PROGRAMMING OPT. CONDITIONS

A pair of vectors (x^*, μ^*) form a primal and dual optimal solution pair if and only if x^* is primal-feasible, μ^* is dual-feasible, and

$$\mu_j^*(b_j - a_j' x^*) = 0, \qquad \forall\, j = 1, \ldots, r. \qquad (*)$$

Proof: If x^* is primal-feasible and μ^* is dual-feasible, then

$$\begin{aligned} b'\mu^* &= \sum_{j=1}^{r} b_j \mu_j^* + \left(c - \sum_{j=1}^{r} a_j \mu_j^* \right)' x^* \\ &= c'x^* + \sum_{j=1}^{r} \mu_j^*(b_j - a_j' x^*) \end{aligned} \qquad (**)$$

So if Eq. (*) holds, we have $b'\mu^* = c'x^*$, and weak duality implies that x^* is primal optimal and μ^* is dual optimal.

Conversely, if (x^*, μ^*) form a primal and dual optimal solution pair, then x^* is primal-feasible, μ^* is dual-feasible, and by the duality theorem, we have $b'\mu^* = c'x^*$. From Eq. (**), we obtain Eq. (*).

CONVEX PROGRAMMING

Consider the problem

$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \quad x \in X, \ g_j(x) \leq 0, \ j = 1, \ldots, r, \end{aligned}$$

where $X \subset \Re^n$ is convex, and $f : X \mapsto \Re$ and $g_j : X \mapsto \Re$ are convex. Assume f^*: finite.

- Recall the connection with the max crossing problem in the MC/MC framework where $M = \text{epi}(p)$ with

$$p(u) = \inf_{x \in X,\, g(x) \leq u} f(x)$$

- Consider the Lagrangian function

$$L(x, \mu) = f(x) + \mu' g(x),$$

the dual function

$$q(\mu) = \begin{cases} \inf_{x \in X} L(x, \mu) & \text{if } \mu \geq 0, \\ -\infty & \text{otherwise} \end{cases}$$

and the dual problem of maximizing $\inf_{x \in X} L(x, \mu)$ over $\mu \geq 0$.

STRONG DUALITY THEOREM

- Assume that f^* is finite, and that one of the following two conditions holds:

  (1) There exists $\overline{x} \in X$ such that $g(\overline{x}) < 0$.

  (2) The functions g_j, $j = 1, \ldots, r$, are affine, and there exists $\overline{x} \in \mathrm{ri}(X)$ such that $g(\overline{x}) \leq 0$.

Then $q^* = f^*$ and the set of optimal solutions of the dual problem is nonempty. Under condition (1) this set is also compact.

- **Proof:** Replace $f(x)$ by $f(x) - f^*$ so that $f(x) - f^* \geq 0$ for all $x \in X$ w/ $g(x) \leq 0$. Apply Nonlinear Farkas' Lemma. Then, there exist $\mu_j^* \geq 0$, s.t.

$$f^* \leq f(x) + \sum_{j=1}^{r} \mu_j^* g_j(x), \qquad \forall\, x \in X$$

- It follows that

$$f^* \leq \inf_{x \in X} \{ f(x) + \mu^{*\prime} g(x) \} \leq \inf_{x \in X,\, g(x) \leq 0} f(x) = f^*.$$

Thus equality holds throughout, and we have

$$f^* = \inf_{x \in X} \left\{ f(x) + \sum_{j=1}^{r} \mu_j^* g_j(x) \right\} = q(\mu^*)$$

QUADRATIC PROGRAMMING DUALITY

- Consider the quadratic program

$$\begin{aligned} &\text{minimize} \quad \tfrac{1}{2}x'Qx + c'x \\ &\text{subject to} \quad Ax \le b, \end{aligned}$$

where Q is positive definite.

- If f^* is finite, then $f^* = q^*$ and there exist both primal and dual optimal solutions, since the constraints are linear.

- Calculation of dual function:

$$q(\mu) = \inf_{x \in \Re^n} \{\tfrac{1}{2}x'Qx + c'x + \mu'(Ax - b)\}$$

The infimum is attained for $x = -Q^{-1}(c + A'\mu)$, and, after substitution and calculation,

$$q(\mu) = -\tfrac{1}{2}\mu' AQ^{-1}A'\mu - \mu'(b + AQ^{-1}c) - \tfrac{1}{2}c'Q^{-1}c$$

- The dual problem, after a sign change, is

$$\begin{aligned} &\text{minimize} \quad \tfrac{1}{2}\mu' P\mu + t'\mu \\ &\text{subject to} \quad \mu \ge 0, \end{aligned}$$

where $P = AQ^{-1}A'$ and $t = b + AQ^{-1}c$.

OPTIMALITY CONDITIONS

• We have $q^* = f^*$, and the vectors x^* and μ^* are optimal solutions of the primal and dual problems, respectively, iff x^* is feasible, $\mu^* \geq 0$, and

$$x^* \in \arg\min_{x \in X} L(x, \mu^*), \qquad \mu_j^* g_j(x^*) = 0, \quad \forall\, j. \tag{1}$$

Proof: If $q^* = f^*$, and x^*, μ^* are optimal, then

$$\begin{aligned} f^* = q^* = q(\mu^*) = \inf_{x \in X} L(x, \mu^*) &\leq L(x^*, \mu^*) \\ &= f(x^*) + \sum_{j=1}^{r} \mu_j^* g_j(x^*) \leq f(x^*), \end{aligned}$$

where the last inequality follows from $\mu_j^* \geq 0$ and $g_j(x^*) \leq 0$ for all j. Hence equality holds throughout above, and (1) holds.

Conversely, if x^*, μ^* are feasible, and (1) holds,

$$\begin{aligned} q(\mu^*) = \inf_{x \in X} L(x, \mu^*) &= L(x^*, \mu^*) \\ &= f(x^*) + \sum_{j=1}^{r} \mu_j^* g_j(x^*) = f(x^*), \end{aligned}$$

so $q^* = f^*$, and x^*, μ^* are optimal. **Q.E.D.**

QUADRATIC PROGRAMMING OPT. COND.

For the quadratic program

$$\begin{aligned} &\text{minimize} \ \ \tfrac{1}{2}x'Qx + c'x \\ &\text{subject to} \ \ Ax \le b, \end{aligned}$$

where Q is positive definite, (x^*, μ^*) is a primal and dual optimal solution pair if and only if:

- Primal and dual feasibility holds:

$$Ax^* \le b, \qquad \mu^* \ge 0$$

- Lagrangian optimality holds [x^* minimizes $L(x, \mu^*)$ over $x \in \Re^n$]. This yields

$$x^* = -Q^{-1}(c + A'\mu^*)$$

- Complementary slackness holds [$(Ax^* - b)'\mu^* = 0$]. It can be written as

$$\mu_j^* > 0 \qquad \Rightarrow \qquad a_j'x^* = b_j, \quad \forall\, j = 1, \ldots, r,$$

where a_j' is the jth row of A, and b_j is the jth component of b.

LINEAR EQUALITY CONSTRAINTS

- The problem is

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in X, \quad g(x) \le 0, \quad Ax = b, \end{array}$$

where X is convex, $g(x) = \big(g_1(x), \ldots, g_r(x)\big)'$, $f : X \mapsto \Re$ and $g_j : X \mapsto \Re$, $j = 1, \ldots, r$, are convex.

- Convert the constraint $Ax = b$ to $Ax \le b$ and $-Ax \le -b$, with corresponding dual variables $\lambda^+ \ge 0$ and $\lambda^- \ge 0$.

- The Lagrangian function is

$$f(x) + \mu' g(x) + (\lambda^+ - \lambda^-)'(Ax - b),$$

and by introducing a dual variable $\lambda = \lambda^+ - \lambda^-$, with no sign restriction, it can be written as

$$L(x, \mu, \lambda) = f(x) + \mu' g(x) + \lambda'(Ax - b).$$

- The dual problem is

$$\begin{array}{ll} \text{maximize} & q(\mu, \lambda) \equiv \inf\limits_{x \in X} L(x, \mu, \lambda) \\ \text{subject to} & \mu \ge 0, \ \lambda \in \Re^m. \end{array}$$

DUALITY AND OPTIMALITY COND.

- **Pure equality constraints:**

  (a) Assume that f^*: finite and there exists $\overline{x} \in \text{ri}(X)$ such that $A\overline{x} = b$. Then $f^* = q^*$ and there exists a dual optimal solution.

  (b) $f^* = q^*$, and (x^*, λ^*) are a primal and dual optimal solution pair if and only if x^* is feasible, and

$$x^* \in \arg\min_{x \in X} L(x, \lambda^*)$$

Note: No complementary slackness for equality constraints.

- **Linear and nonlinear constraints:**

  (a) Assume f^*: finite, that there exists $\overline{x} \in X$ such that $A\overline{x} = b$ and $g(\overline{x}) < 0$, and that there exists $\tilde{x} \in \text{ri}(X)$ such that $A\tilde{x} = b$. Then $q^* = f^*$ and there exists a dual optimal solution.

  (b) $f^* = q^*$, and (x^*, μ^*, λ^*) are a primal and dual optimal solution pair if and only if x^* is feasible, $\mu^* \geq 0$, and

$$x^* \in \arg\min_{x \in X} L(x, \mu^*, \lambda^*), \;\; \mu_j^* g_j(x^*) = 0, \quad \forall\, j$$