LECTURE 6

LECTURE OUTLINE

- Nonemptiness of closed set intersections
  - Simple version
  - More complex version
- Existence of optimal solutions
- Preservation of closure under linear transformation
- Hyperplanes

ROLE OF CLOSED SET INTERSECTIONS I

- **A fundamental question:** Given a sequence of nonempty closed sets $\{C_k\}$ in $\Re^n$ with $C_{k+1} \subset C_k$ for all k, when is $\cap_{k=0}^{\infty} C_k$ nonempty?

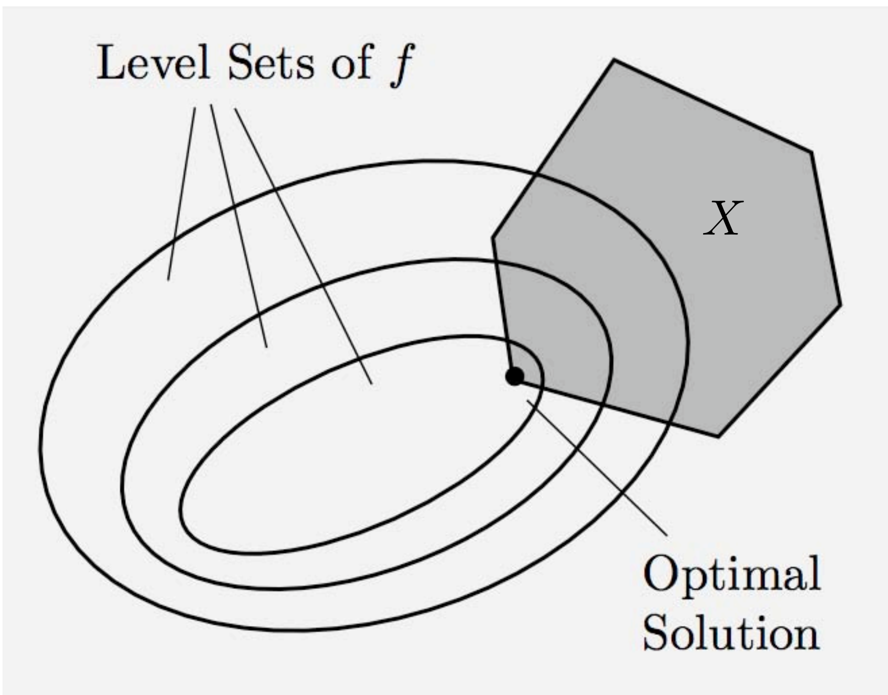
- Set intersection theorems are significant in at least three major contexts, which we will discuss in what follows:

Does a function $f : \Re^n \mapsto (-\infty, \infty]$ attain a minimum over a set X?

This is true if and only if

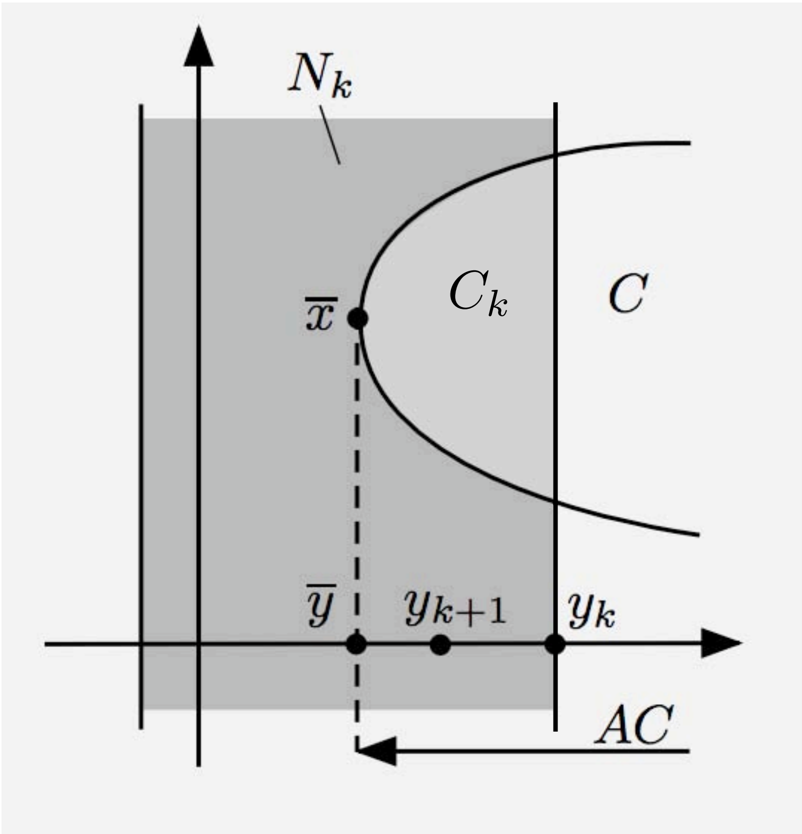
$$\text{Intersection of nonempty } \big\{x \in X \mid f(x) \leq \gamma_k\big\}$$

is nonempty.



ROLE OF CLOSED SET INTERSECTIONS II

If C is closed and A is a matrix, is AC closed?



- **If C_1 and C_2 are closed, is $C_1 + C_2$ closed?**
  - This is a special case.
  - Write
$$C_1 + C_2 = A(C_1 \times C_2),$$
where $A(x_1, x_2) = x_1 + x_2$.

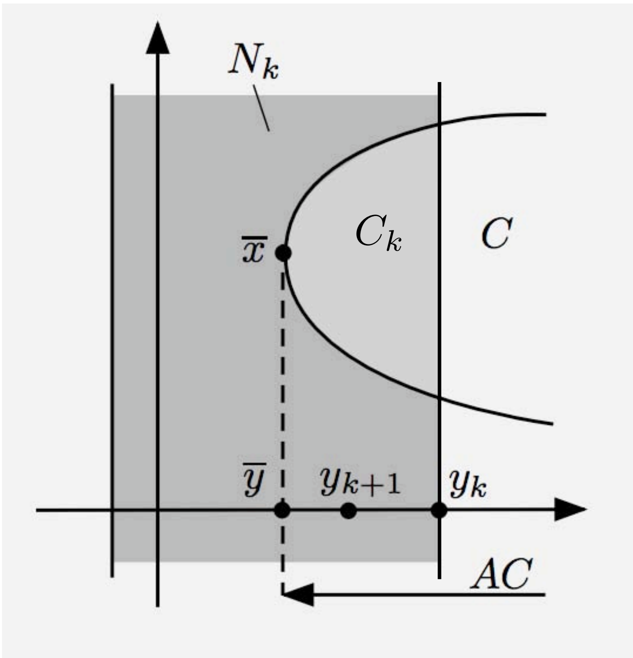
CLOSURE UNDER LINEAR TRANSFORMATION

- Let C be a nonempty closed convex, and let A be a matrix with nullspace $N(A)$. Then $A\,C$ is closed if $R_C \cap N(A) = \{0\}$.

Proof: Let $\{y_k\} \subset A\,C$ with $y_k \to \overline{y}$. Define the nested sequence $C_k = C \cap N_k$, where

$$N_k = \{x \mid Ax \in W_k\}, \quad W_k = \big\{z \mid \|z-\overline{y}\| \leq \|y_k-\overline{y}\|\big\}$$

We have $R_{N_k} = N(A)$, so C_k is compact, and $\{C_k\}$ has nonempty intersection. **Q.E.D.**



- **A special case:** $C_1 + C_2$ is closed if C_1, C_2 are closed and one of the two is compact. [Write $C_1+C_2 = A(C_1 \times C_2)$, where $A(x_1, x_2) = x_1+x_2$.]

- **Related theorem:** AX is closed if X is polyhedral. To be shown later by a more refined method.

ROLE OF CLOSED SET INTERSECTIONS III

- Let $F : \Re^{n+m} \mapsto (-\infty, \infty]$ be a closed proper convex function, and consider

$$f(x) = \inf_{z \in \Re^m} F(x, z)$$

- **If $F(x, z)$ is closed, is $f(x)$ closed?**
  - Critical question in duality theory.
- **1st fact:** If F is convex, then f is also convex.
- **2nd fact:**

$$P\big(\text{epi}(F)\big) \subset \text{epi}(f) \subset \text{cl}\Big(P\big(\text{epi}(F)\big)\Big),$$

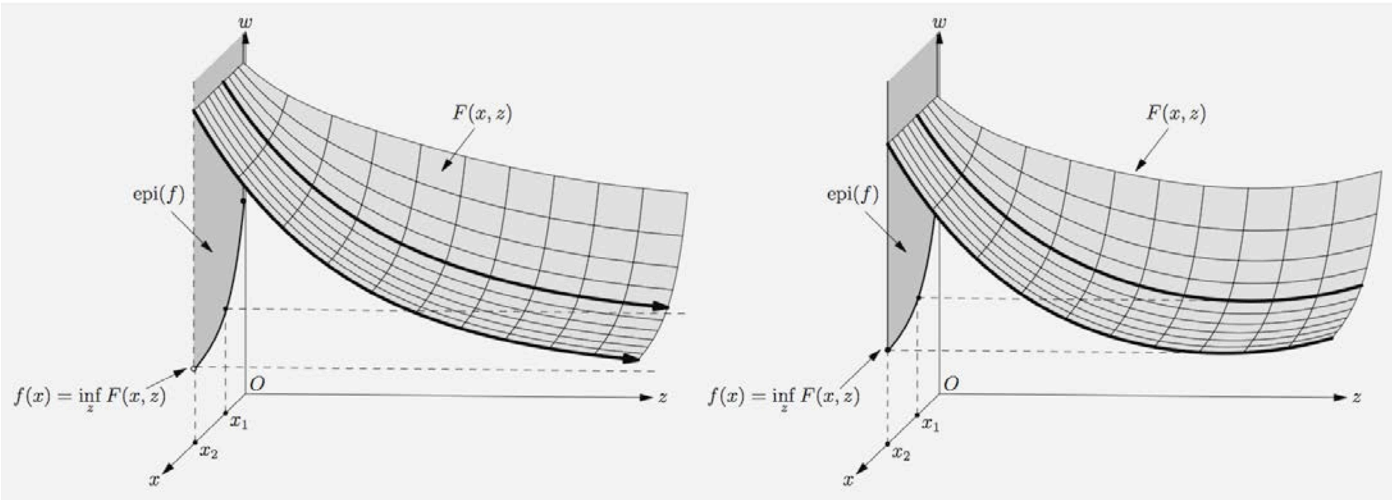
where $P(\cdot)$ denotes projection on the space of (x, w), i.e., for any subset S of $\Re^{n+m+1}$, $P(S) = \big\{(x, w) \mid (x, z, w) \in S\big\}$.

- Thus, if F is closed and there is structure guaranteeing that the projection preserves closedness, then f is closed.

- ... but convexity and closedness of F does not guarantee closedness of f.

PARTIAL MINIMIZATION: VISUALIZATION

- Connection of preservation of closedness under partial minimization and attainment of infimum over z for fixed x.



- **Counterexample:** Let

$$F(x,z) = \begin{cases} e^{-\sqrt{xz}} & \text{if } x \geq 0,\ z \geq 0, \\ \infty & \text{otherwise.} \end{cases}$$

- F convex and closed, but

$$f(x) = \inf_{z \in \Re} F(x,z) = \begin{cases} 0 & \text{if } x > 0, \\ 1 & \text{if } x = 0, \\ \infty & \text{if } x < 0, \end{cases}$$

is not closed.

PARTIAL MINIMIZATION THEOREM

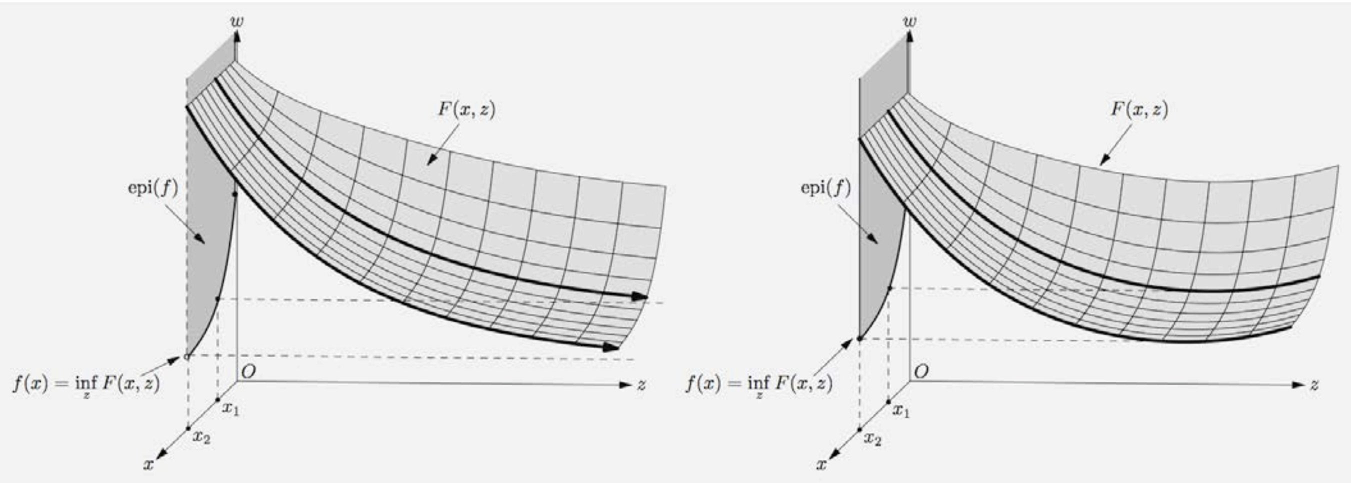
• Let $F : \Re^{n+m} \mapsto (-\infty, \infty]$ be a closed proper convex function, and consider $f(x) = \inf_{z \in \Re^m} F(x, z)$.

• Every set intersection theorem yields a closedness result. The simplest case is the following:

• **Preservation of Closedness Under Compactness:** If there exist $\overline{x} \in \Re^n$, $\overline{\gamma} \in \Re$ such that the set

$$\{z \mid F(\overline{x}, z) \leq \overline{\gamma}\}$$

is nonempty and compact, then f is convex, closed, and proper. Also, for each $x \in \text{dom}(f)$, the set of minima of $F(x, \cdot)$ is nonempty and compact.



MORE REFINED ANALYSIS - A SUMMARY

- We noted that there is a common mathematical root to three basic questions:
  - Existence of of solutions of convex optimization problems
  - Preservation of closedness of convex sets under a linear transformation
  - Preservation of closedness of convex functions under partial minimization
- The common root is the question of nonemptiness of intersection of a nested sequence of closed sets
- The preceding development in this lecture resolved this question by assuming that all the sets in the sequence are compact
- A more refined development makes instead various assumptions about the directions of recession and the lineality space of the sets in the sequence
- Once the appropriately refined set intersection theory is developed, sharper results relating to the three questions can be obtained
- The remaining slides up to hyperplanes summarize this development as an aid for self-study using Sections 1.4.2, 1.4.3, and Sections 3.2, 3.3

ASYMPTOTIC SEQUENCES

- Given nested sequence $\{C_k\}$ of closed convex sets, $\{x_k\}$ is an *asymptotic sequence* if

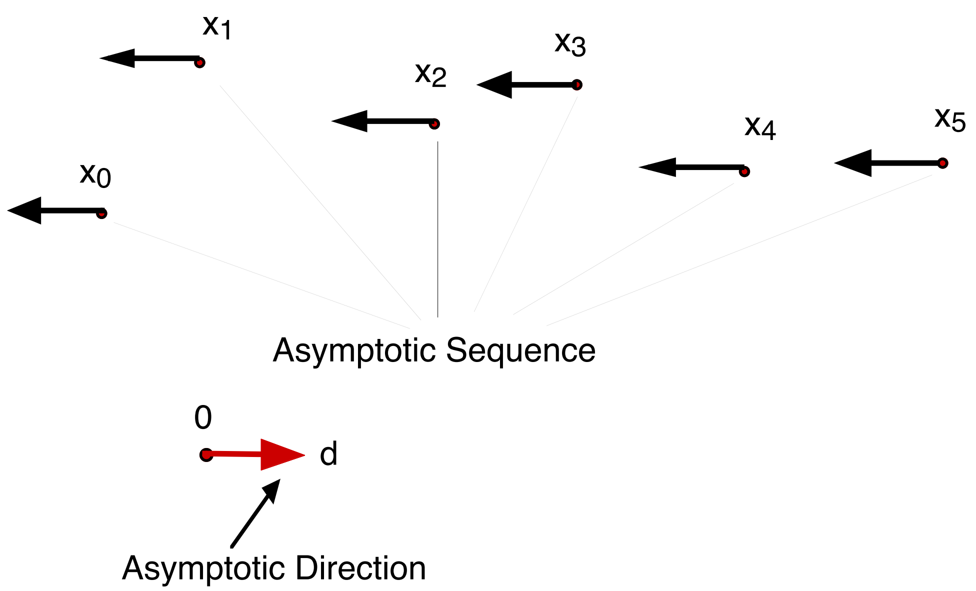
$$x_k \in C_k, \qquad x_k \neq 0, \qquad k = 0, 1, \ldots$$

$$\|x_k\| \to \infty, \qquad \frac{x_k}{\|x_k\|} \to \frac{d}{\|d\|}$$

where d is a nonzero common direction of recession of the sets C_k.

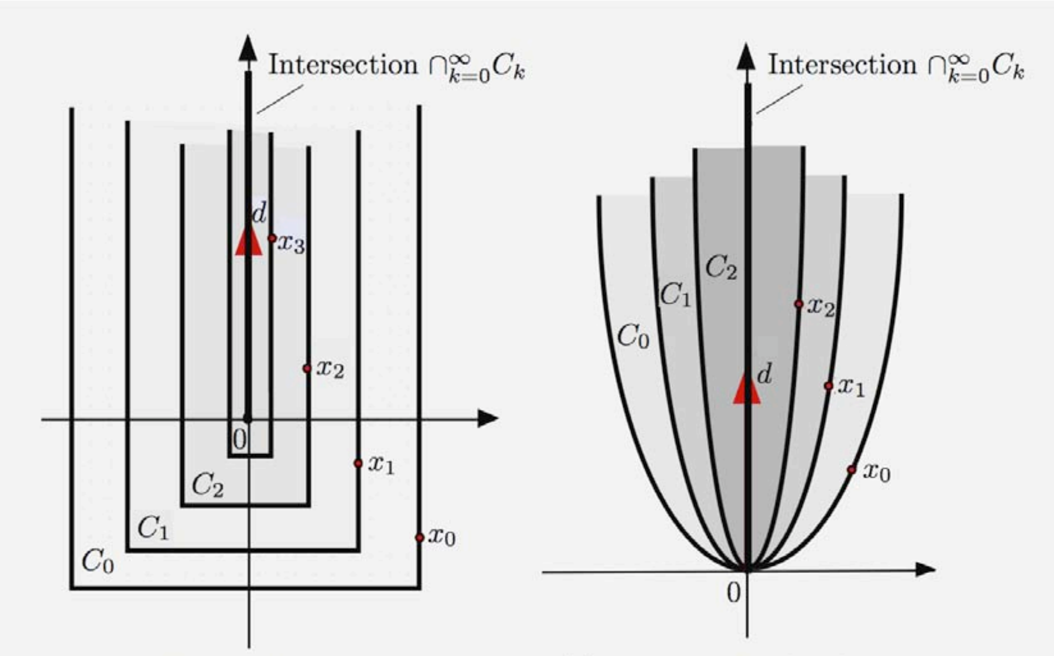
- As a special case we define asymptotic sequence of a closed convex set C (use $C_k \equiv C$).
- Every unbounded $\{x_k\}$ with $x_k \in C_k$ has an asymptotic subsequence.
- $\{x_k\}$ is called *retractive* if for some $\bar{k}$, we have

$$x_k - d \in C_k, \quad \forall\, k \geq \bar{k}.$$



RETRACTIVE SEQUENCES

- A nested sequence $\{C_k\}$ of closed convex sets is *retractive* if all its asymptotic sequences are retractive.



(a) Retractive Set Sequence (b) Nonretractive Set Sequence

- A closed halfspace (viewed as a sequence with identical components) is retractive.

- Intersections and Cartesian products of retractive set sequences are retractive.

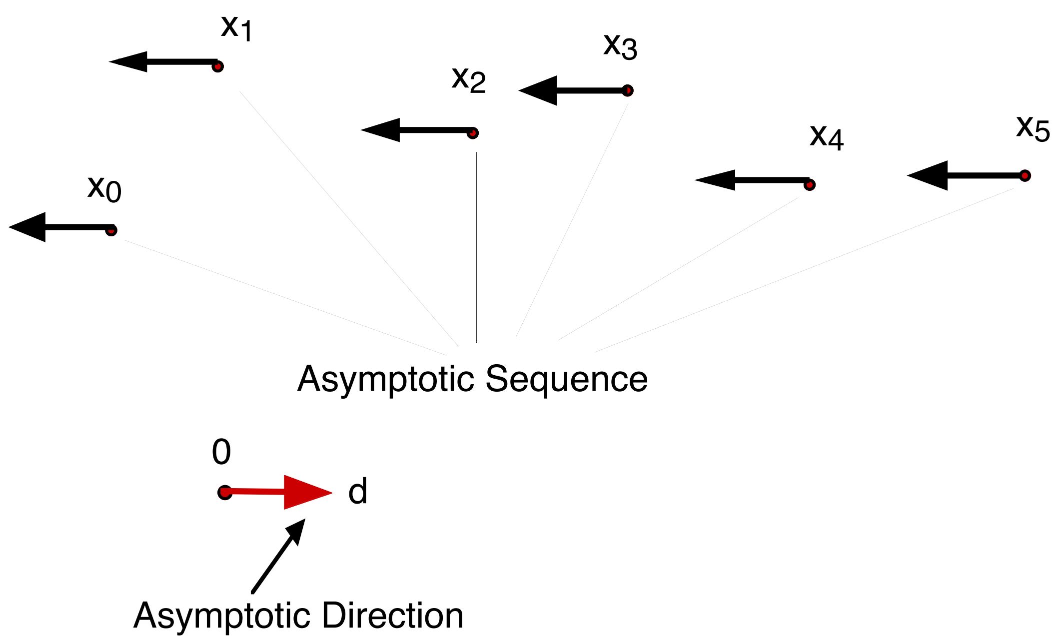
- A polyhedral set is retractive. Also the vector sum of a convex compact set and a retractive convex set is retractive.

- Nonpolyhedral cones and level sets of quadratic functions need not be retractive.

SET INTERSECTION THEOREM I

Proposition: If $\{C_k\}$ is retractive, then $\cap_{k=0}^{\infty} C_k$ is nonempty.

- Key proof ideas:
  (a) The intersection $\cap_{k=0}^{\infty} C_k$ is empty iff the sequence $\{x_k\}$ of minimum norm vectors of C_k is unbounded (so a subsequence is asymptotic).
  (b) An asymptotic sequence $\{x_k\}$ of minimum norm vectors cannot be retractive, because such a sequence eventually gets closer to 0 when shifted opposite to the asymptotic direction.



SET INTERSECTION THEOREM II

Proposition: Let $\{C_k\}$ be a nested sequence of nonempty closed convex sets, and X be a retractive set such that all the sets $\overline{C}_k = X \cap C_k$ are nonempty. Assume that

$$R_X \cap R \subset L,$$

where

$$R = \cap_{k=0}^{\infty} R_{C_k}, \qquad L = \cap_{k=0}^{\infty} L_{C_k}$$

Then $\{\overline{C}_k\}$ is retractive and $\cap_{k=0}^{\infty} \overline{C}_k$ is nonempty.

- Special cases:
  - $X = \Re^n$, $R = L$ ("cylindrical" sets C_k)
  - $R_X \cap R = \{0\}$ (no nonzero common recession direction of X and $\cap_k C_k$)

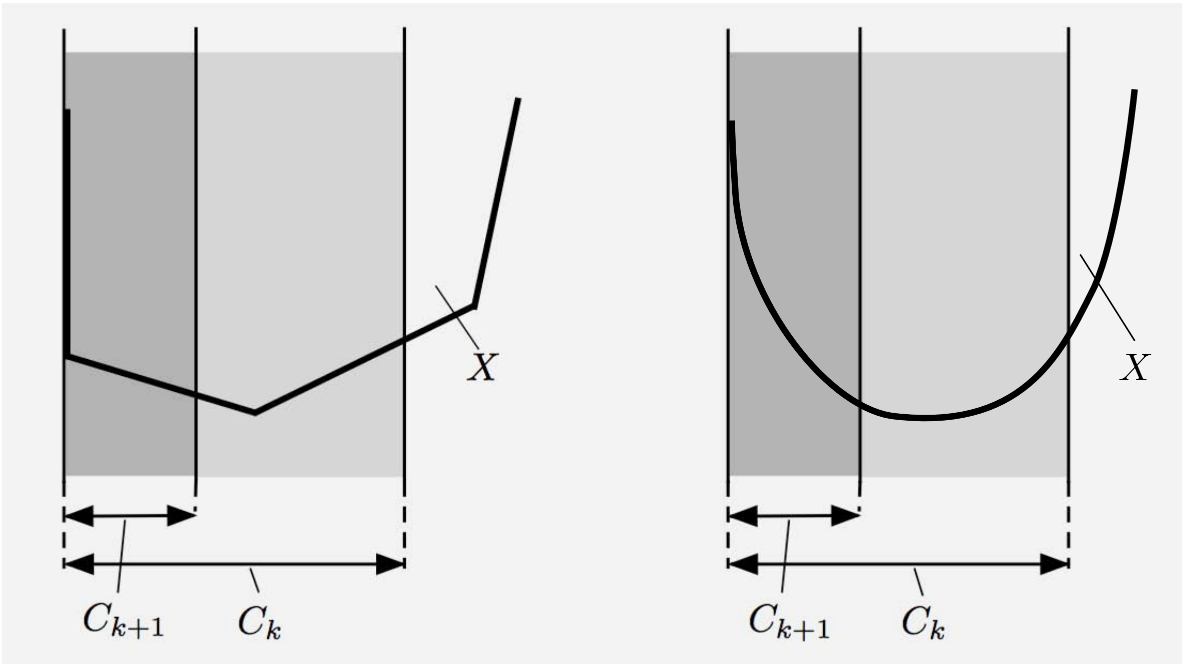
Proof: The set of common directions of recession of $\overline{C}_k$ is $R_X \cap R$. For any asymptotic sequence $\{x_k\}$ corresponding to $d \in R_X \cap R$:

(1) $x_k - d \in C_k$ (because $d \in L$)

(2) $x_k - d \in X$ (because X is retractive)

So $\{\overline{C}_k\}$ is retractive.

NEED TO ASSUME THAT X IS RETRACTIVE



Consider $\cap_{k=0}^{\infty} \overline{C}_k$, with $\overline{C}_k = X \cap C_k$

- The condition $R_X \cap R \subset L$ holds
- In the figure on the left, X is polyhedral.
- In the figure on the right, X is nonpolyhedral and nonretrative, and

$$\cap_{k=0}^{\infty} \overline{C}_k = \emptyset$$

LINEAR AND QUADRATIC PROGRAMMING

- **Theorem:** Let

$$f(x) = x'Qx + c'x, \quad X = \{x \mid a_j'x + b_j \leq 0,\ j = 1, \ldots, r\}$$

where Q is symmetric positive semidefinite. If the minimal value of f over X is finite, there exists a minimum of f over X.

Proof: (Outline) Write

$$\text{Set of Minima} = \cap_{k=0}^{\infty}\big(X \cap \{x \mid x'Qx + c'x \leq \gamma_k\}\big)$$

with

$$\gamma_k \downarrow f^* = \inf_{x \in X} f(x).$$

Verify the condition $R_X \cap R \subset L$ of the preceding set intersection theorem, where R and L are the sets of common recession and lineality directions of the sets

$$\{x \mid x'Qx + c'x \leq \gamma_k\}$$

Q.E.D.

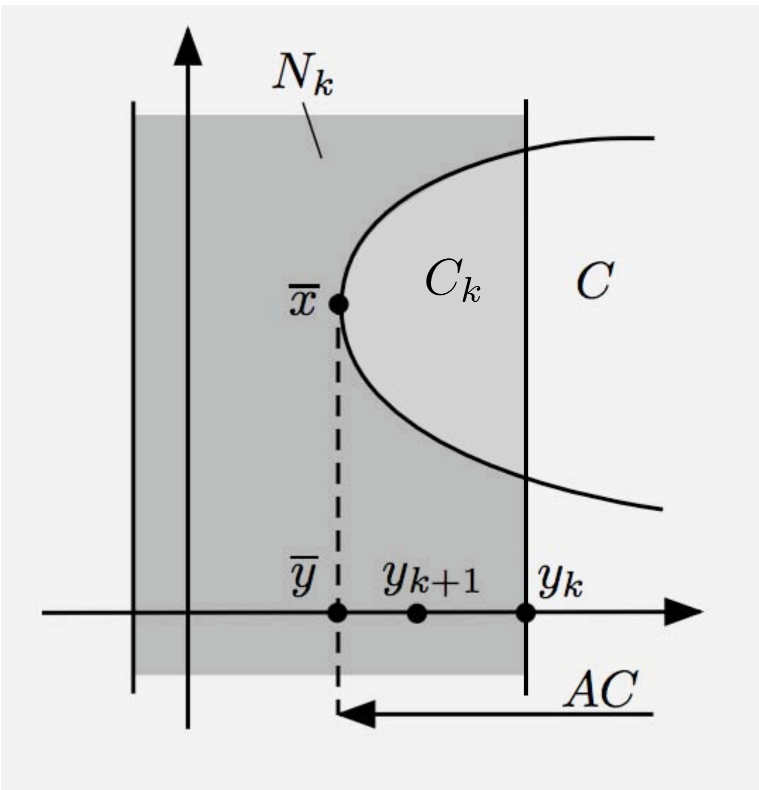
CLOSURE UNDER LINEAR TRANSFORMATIONS

- Let C be a nonempty closed convex, and let A be a matrix with nullspace $N(A)$.
  - (a) AC is closed if $R_C \cap N(A) \subset L_C$.
  - (b) $A(X \cap C)$ is closed if X is a retractive set and

$$R_X \cap R_C \cap N(A) \subset L_C,$$

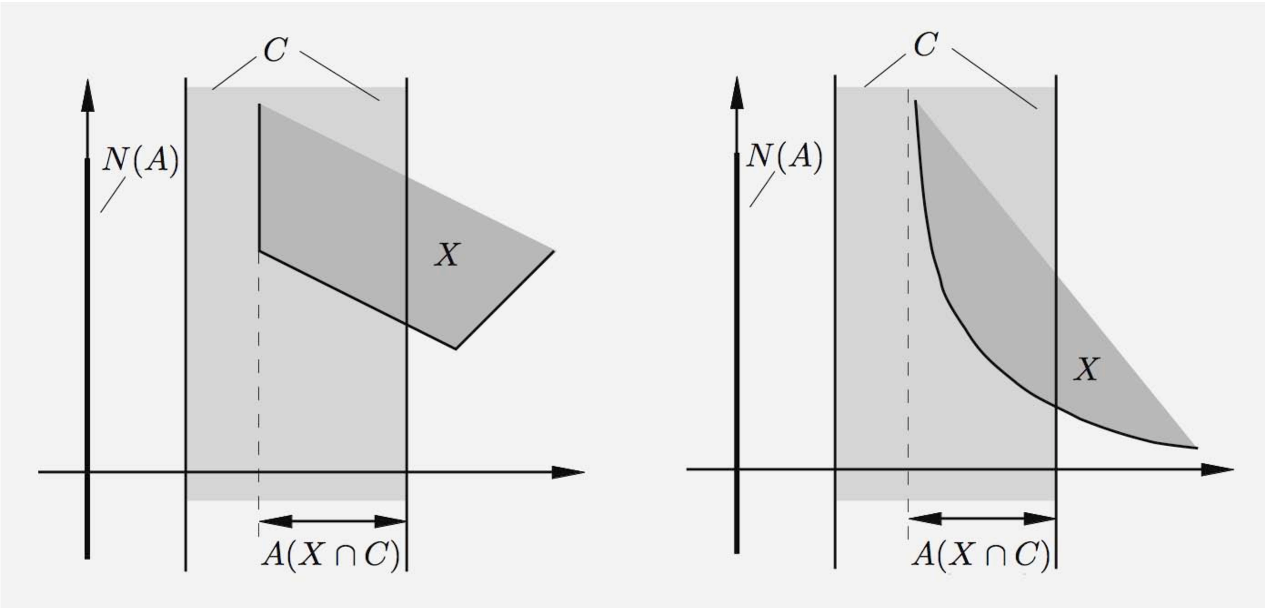
Proof: (Outline) Let $\{y_k\} \subset AC$ with $y_k \to \overline{y}$. We prove $\cap_{k=0}^{\infty} C_k \neq \emptyset$, where $C_k = C \cap N_k$, and

$$N_k = \{x \mid Ax \in W_k\}, \quad W_k = \left\{z \mid \|z - \overline{y}\| \leq \|y_k - \overline{y}\|\right\}$$



- **Special Case:** AX is closed if X is polyhedral.

NEED TO ASSUME THAT X IS RETRACTIVE



Consider closedness of $A(X \cap C)$

- In both examples the condition

$$R_X \cap R_C \cap N(A) \subset L_C$$

is satisfied.

- However, in the example on the right, X is not retractive, and the set $A(X \cap C)$ is not closed.

CLOSEDNESS OF VECTOR SUMS

- Let $C_1, \ldots, C_m$ be nonempty closed convex subsets of $\Re^n$ such that the equality $d_1 + \cdots + d_m = 0$ for some vectors $d_i \in R_{C_i}$ implies that $d_i = 0$ for all $i = 1, \ldots, m$. Then $C_1 + \cdots + C_m$ is a closed set.

- **Special Case:** If C_1 and $-C_2$ are closed convex sets, then $C_1 - C_2$ is closed if $R_{C_1} \cap R_{C_2} = \{0\}$.

Proof: The Cartesian product $C = C_1 \times \cdots \times C_m$ is closed convex, and its recession cone is $R_C = R_{C_1} \times \cdots \times R_{C_m}$. Let A be defined by

$$A(x_1, \ldots, x_m) = x_1 + \cdots + x_m$$

Then

$$A\,C = C_1 + \cdots + C_m,$$

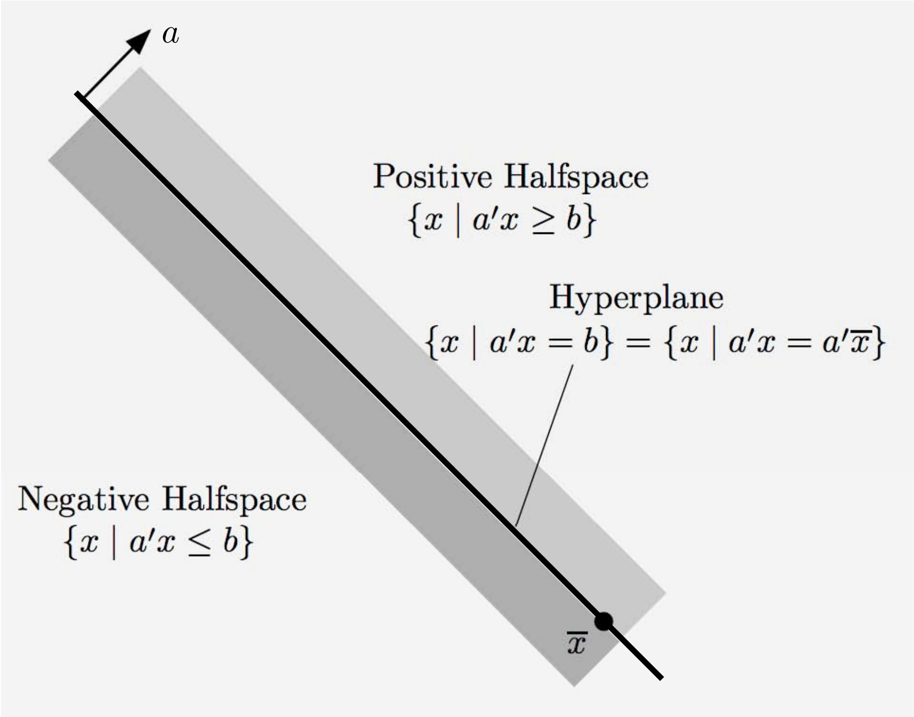
and

$$N(A) = \Big\{(d_1, \ldots, d_m) \mid d_1 + \cdots + d_m = 0\Big\}$$

$$R_C \cap N(A) = \Big\{(d_1, \ldots, d_m) \mid d_1 + \cdots + d_m = 0,\ d_i \in R_{C_i},\ \forall\, i\Big\}$$

By the given condition, $R_C \cap N(A) = \{0\}$, so $A\,C$ is closed. **Q.E.D.**

HYPERPLANES



- A *hyperplane* is a set of the form $\{x \mid a'x = b\}$, where a is nonzero vector in $\Re^n$ and b is a scalar.

- We say that two sets C_1 and C_2 are *separated by a hyperplane* $H = \{x \mid a'x = b\}$ if each lies in a different closed halfspace associated with H, i.e.,

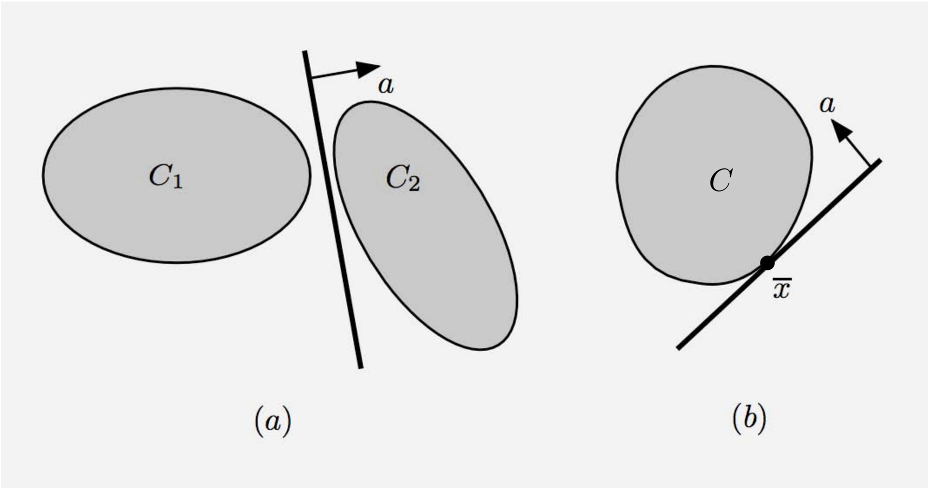
$$\text{either} \quad a'x_1 \leq b \leq a'x_2, \qquad \forall\, x_1 \in C_1,\ \forall\, x_2 \in C_2,$$

$$\text{or} \quad a'x_2 \leq b \leq a'x_1, \qquad \forall\, x_1 \in C_1,\ \forall\, x_2 \in C_2$$

- If $\overline{x}$ belongs to the closure of a set C, a hyperplane that separates C and the singleton set $\{\overline{x}\}$ is said be *supporting* C *at* $\overline{x}$.

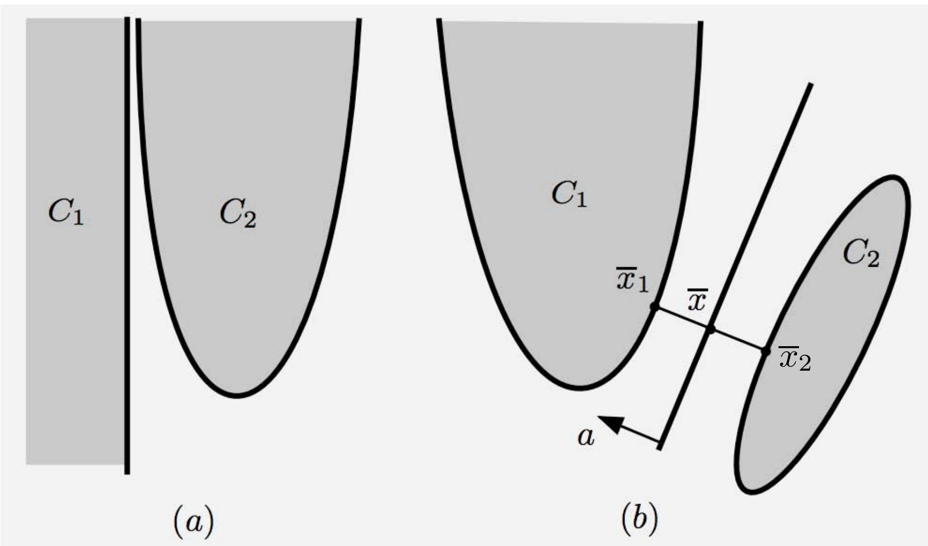
VISUALIZATION

- Separating and supporting hyperplanes:



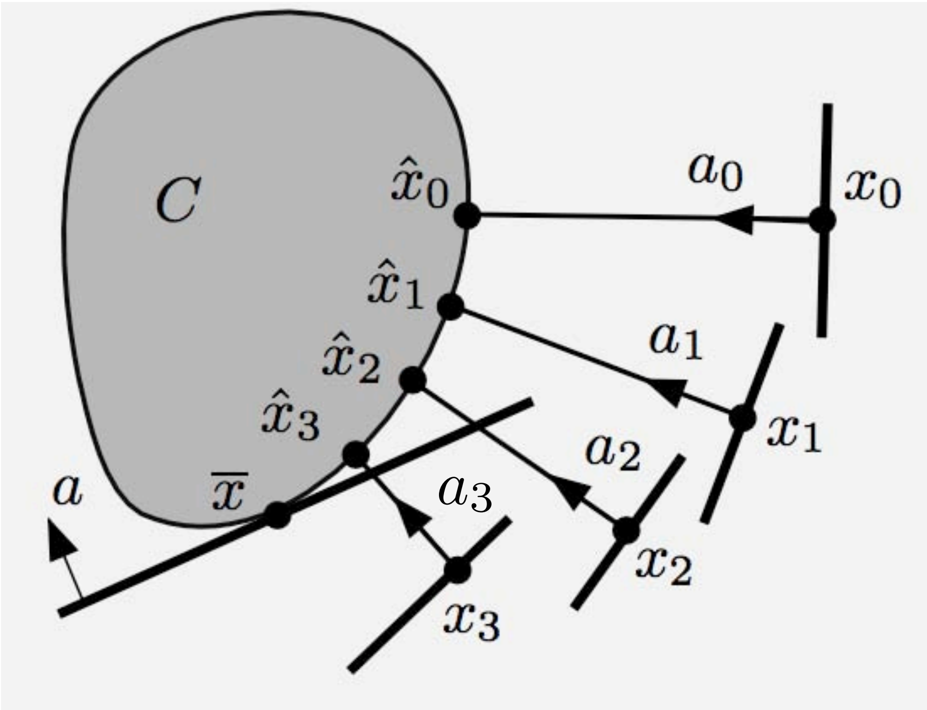
- A separating $\{x \mid a'x = b\}$ that is disjoint from C_1 and C_2 is called *strictly* separating:

$$a'x_1 < b < a'x_2, \qquad \forall\ x_1 \in C_1,\ \forall\ x_2 \in C_2$$



SUPPORTING HYPERPLANE THEOREM

- Let C be convex and let $\overline{x}$ be a vector that is not an interior point of C. Then, there exists a hyperplane that passes through $\overline{x}$ and contains C in one of its closed halfspaces.



Proof: Take a sequence $\{x_k\}$ that does not belong to $\mathrm{cl}(C)$ and converges to $\overline{x}$. Let $\hat{x}_k$ be the projection of x_k on $\mathrm{cl}(C)$. We have for all $x \in \mathrm{cl}(C)$

$$a_k'x \geq a_k'x_k, \qquad \forall\ x \in \mathrm{cl}(C),\ \forall\ k = 0, 1, \ldots,$$

where $a_k = (\hat{x}_k - x_k)/\|\hat{x}_k - x_k\|$. Let a be a limit point of $\{a_k\}$, and take limit as $k \to \infty$. **Q.E.D.**

SEPARATING HYPERPLANE THEOREM

- Let C_1 and C_2 be two nonempty convex subsets of $\Re^n$. If C_1 and C_2 are disjoint, there exists a hyperplane that separates them, i.e., there exists a vector $a \neq 0$ such that

$$a'x_1 \leq a'x_2, \qquad \forall\, x_1 \in C_1,\ \forall\, x_2 \in C_2.$$

Proof: Consider the convex set

$$C_1 - C_2 = \{x_2 - x_1 \mid x_1 \in C_1,\, x_2 \in C_2\}$$

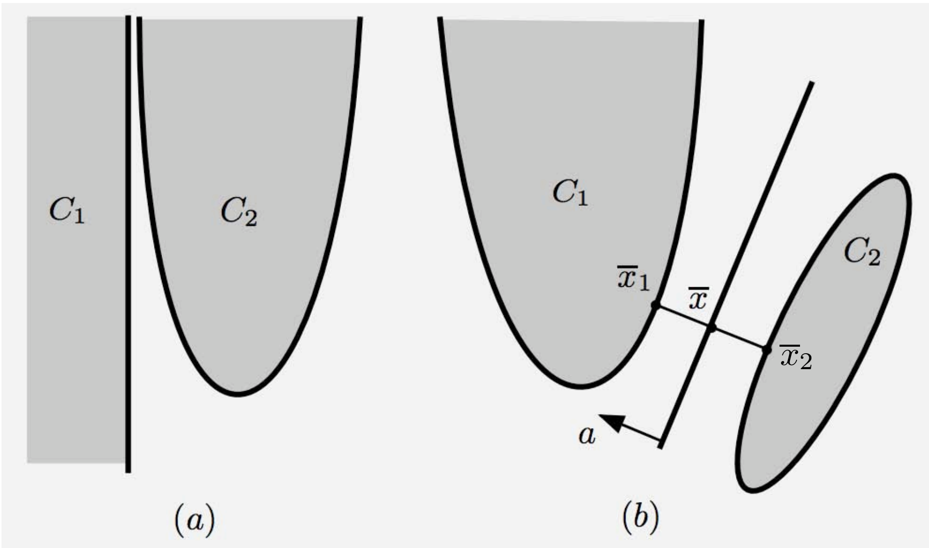
Since C_1 and C_2 are disjoint, the origin does not belong to $C_1 - C_2$, so by the Supporting Hyperplane Theorem, there exists a vector $a \neq 0$ such that

$$0 \leq a'x, \qquad \forall\, x \in C_1 - C_2,$$

which is equivalent to the desired relation. **Q.E.D.**

STRICT SEPARATION THEOREM

- **Strict Separation Theorem**: Let C_1 and C_2 be two disjoint nonempty convex sets. If C_1 is closed, and C_2 is compact, there exists a hyperplane that strictly separates them.



Proof: (Outline) Consider the set $C_1 - C_2$. Since C_1 is closed and C_2 is compact, $C_1 - C_2$ is closed. Since $C_1 \cap C_2 = \emptyset$, $0 \notin C_1 - C_2$. Let $\overline{x}_1 - \overline{x}_2$ be the projection of 0 onto $C_1 - C_2$. The strictly separating hyperplane is constructed as in (b).

- **Note**: Any conditions that guarantee closedness of $C_1 - C_2$ guarantee existence of a strictly separating hyperplane. However, there may exist a strictly separating hyperplane without $C_1 - C_2$ being closed.