LECTURE 5

LECTURE OUTLINE

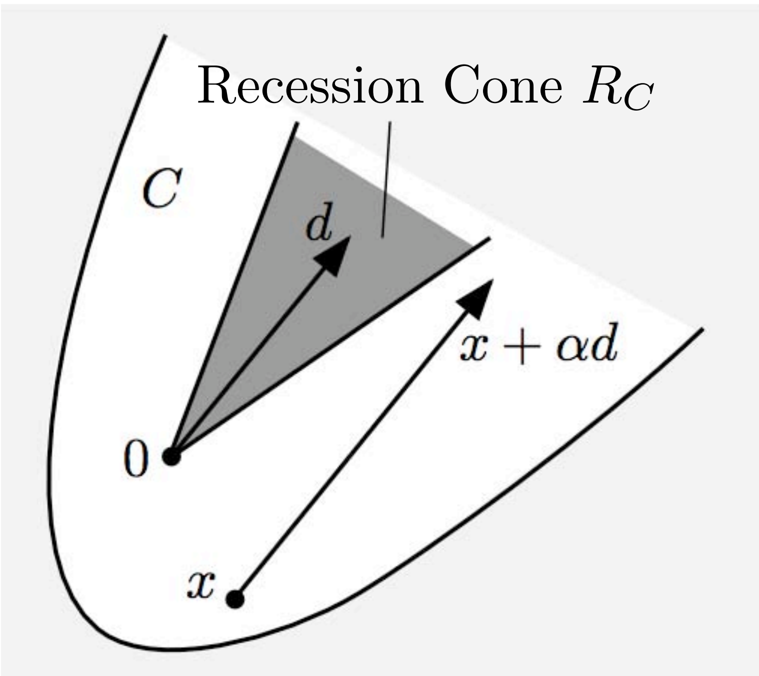
- Recession cones and lineality space
- Directions of recession of convex functions
- Local and global minima
- Existence of optimal solutions

Reading: Section 1.4, 3.1, 3.2

RECESSION CONE OF A CONVEX SET

- Given a nonempty convex set C, a vector d is a *direction of recession* if starting at **any** x in C and going indefinitely along d, we never cross the relative boundary of C to points outside C:

$$x + \alpha d \in C, \qquad \forall\, x \in C, \;\; \forall\, \alpha \geq 0$$



- *Recession cone* of C (denoted by R_C): The set of all directions of recession.

- R_C is a cone containing the origin.

RECESSION CONE THEOREM

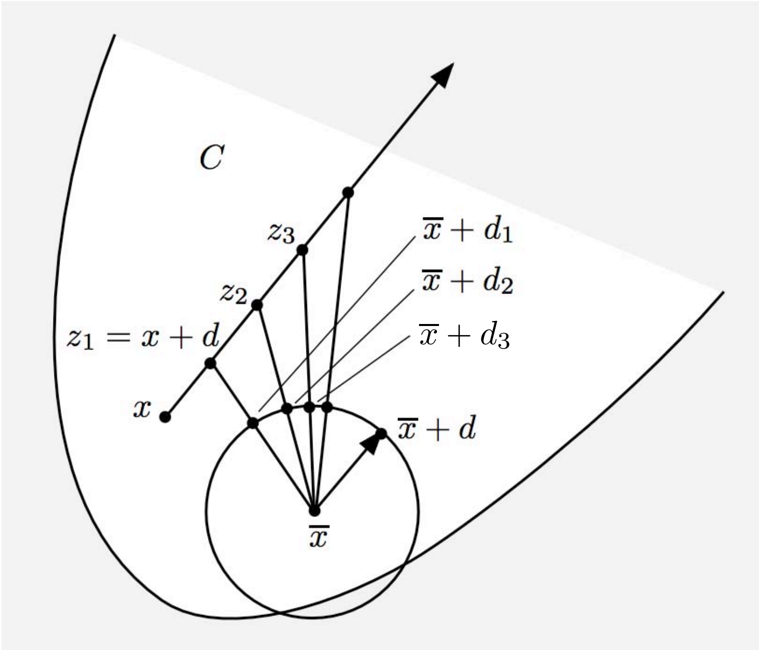
- Let C be a nonempty closed convex set.
  (a) The recession cone R_C is a closed convex cone.
  (b) A vector d belongs to R_C if and only if there exists *some* vector $x \in C$ such that $x + \alpha d \in C$ for all $\alpha \geq 0$.
  (c) R_C contains a nonzero direction if and only if C is unbounded.
  (d) The recession cones of C and $\mathrm{ri}(C)$ are equal.
  (e) If D is another closed convex set such that $C \cap D \neq \emptyset$, we have

$$R_{C \cap D} = R_C \cap R_D$$

More generally, for any collection of closed convex sets C_i, $i \in I$, where I is an arbitrary index set and $\cap_{i \in I} C_i$ is nonempty, we have

$$R_{\cap_{i \in I} C_i} = \cap_{i \in I} R_{C_i}$$

PROOF OF PART (B)



- Let $d \neq 0$ be such that there exists a vector $x \in C$ with $x + \alpha d \in C$ for all $\alpha \geq 0$. We fix $\overline{x} \in C$ and $\alpha > 0$, and we show that $\overline{x} + \alpha d \in C$. By scaling d, it is enough to show that $\overline{x} + d \in C$.

For $k = 1, 2, \ldots$, let

$$z_k = x + kd, \qquad d_k = \frac{(z_k - \overline{x})}{\|z_k - \overline{x}\|}\|d\|$$

We have

$$\frac{d_k}{\|d\|} = \frac{\|z_k - x\|}{\|z_k - \overline{x}\|}\frac{d}{\|d\|} + \frac{x - \overline{x}}{\|z_k - \overline{x}\|}, \quad \frac{\|z_k - x\|}{\|z_k - \overline{x}\|} \to 1, \quad \frac{x - \overline{x}}{\|z_k - \overline{x}\|} \to 0,$$

so $d_k \to d$ and $\overline{x} + d_k \to \overline{x} + d$. Use the convexity and closedness of C to conclude that $\overline{x} + d \in C$.

LINEALITY SPACE

- The *lineality space* of a convex set C, denoted by L_C, is the subspace of vectors d such that $d \in R_C$ and $-d \in R_C$:

$$L_C = R_C \cap (-R_C)$$

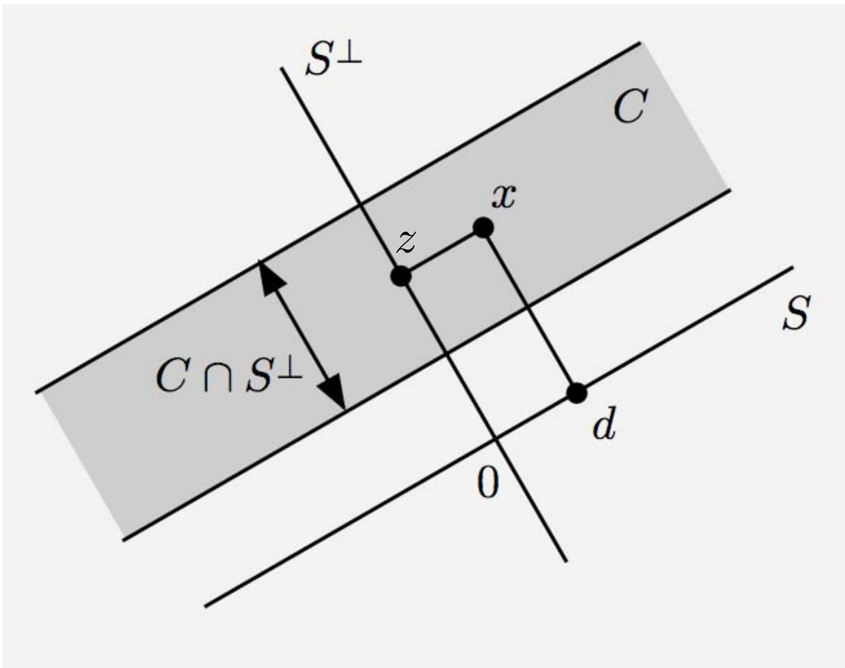
- If $d \in L_C$, the entire line defined by d is contained in C, starting at any point of C.

- *Decomposition of a Convex Set*: Let C be a nonempty convex subset of $\Re^n$. Then,

$$C = L_C + (C \cap L_C^\perp).$$

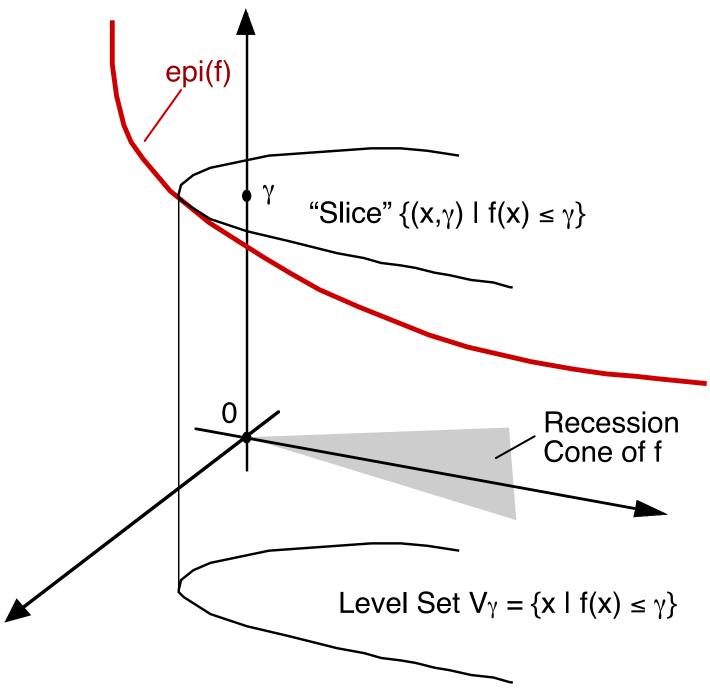
- Allows us to prove properties of C on $C \cap L_C^\perp$ and extend them to C.

- True also if L_C is replaced by a subspace $S \subset L_C$.



DIRECTIONS OF RECESSION OF A FN

- We aim to characterize directions of monotonic decrease of convex functions.
- Some basic geometric observations:
  - The "horizontal directions" in the recession cone of the epigraph of a convex function f are directions along which the level sets are unbounded.
  - Along these directions the level sets $\{x \mid f(x) \leq \gamma\}$ are unbounded and f is monotonically nondecreasing.
- These are the *directions of recession* of f.



RECESSION CONE OF LEVEL SETS

- *Proposition*: Let $f : \Re^n \mapsto (-\infty, \infty]$ be a closed proper convex function and consider the level sets $V_\gamma = \{x \mid f(x) \leq \gamma\}$, where γ is a scalar. Then:

  (a) All the nonempty level sets V_γ have the same recession cone:

$$R_{V_\gamma} = \{d \mid (d, 0) \in R_{\mathrm{epi}(f)}\}$$

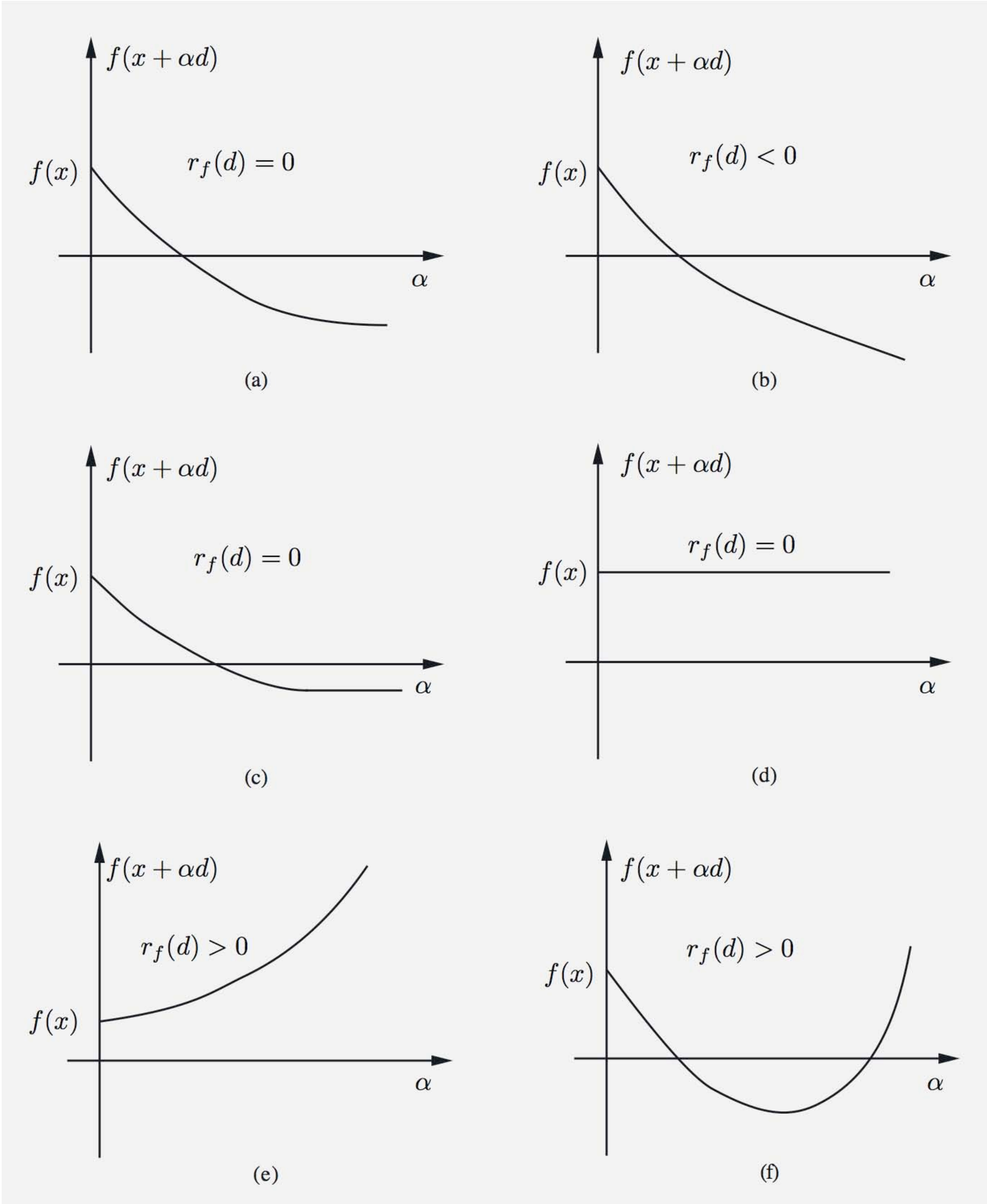
  (b) If one nonempty level set V_γ is compact, then all level sets are compact.

Proof: (a) Just translate to math the fact that

R_{V_γ} = the "horizontal" directions of recession of epi(f)

(b) Follows from (a).

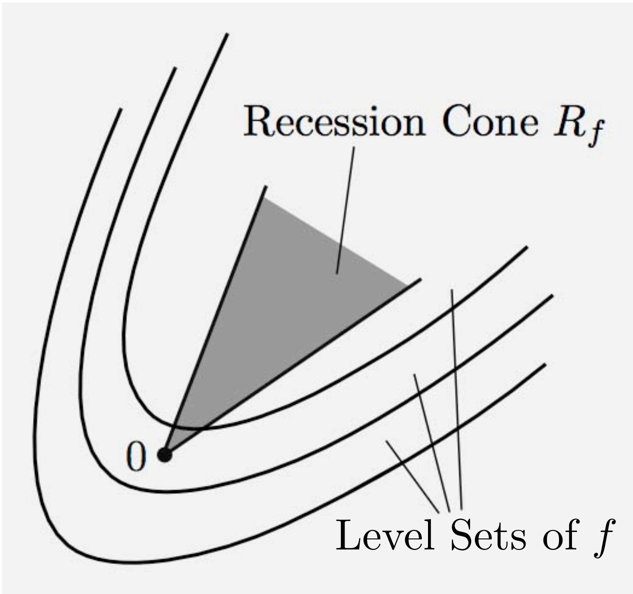
DESCENT BEHAVIOR OF A CONVEX FN



- y is a direction of recession in (a)-(d).
- This behavior is *independent of the starting point* x, as long as $x \in \text{dom}(f)$.

RECESSION CONE OF A CONVEX FUNCTION

- For a closed proper convex function $f : \Re^n \mapsto (-\infty, \infty]$, the (common) recession cone of the nonempty level sets $V_\gamma = \{x \mid f(x) \leq \gamma\}$, $\gamma \in \Re$, is the *recession cone of* f, and is denoted by R_f.



- Terminology:
  - $d \in R_f$: a *direction of recession of* f.
  - $L_f = R_f \cap (-R_f)$: the *lineality space of* f.
  - $d \in L_f$: a *direction of constancy* of f.
- **Example:** For the pos. semidefinite quadratic

$$f(x) = x'Qx + a'x + b,$$

the recession cone and constancy space are

$$R_f = \{d \mid Qd = 0,\ a'd \leq 0\},\ \ L_f = \{d \mid Qd = 0,\ a'd = 0\}$$

RECESSION FUNCTION

- Function $r_f : \Re^n \mapsto (-\infty, \infty]$ whose epigraph is $R_{\mathrm{epi}(f)}$ is the *recession function* of f.
- Characterizes the recession cone:

$$R_f = \big\{d \mid r_f(d) \le 0\big\}, \quad L_f = \big\{d \mid r_f(d) = r_f(-d) = 0\big\}$$

since $R_f = \{(d, 0) \in R_{\mathrm{epi}(f)}\}$.

- Can be shown that

$$r_f(d) = \sup_{\alpha > 0} \frac{f(x + \alpha d) - f(x)}{\alpha} = \lim_{\alpha \to \infty} \frac{f(x + \alpha d) - f(x)}{\alpha}$$

- Thus $r_f(d)$ is the "asymptotic slope" of f in the direction d. In fact,

$$r_f(d) = \lim_{\alpha \to \infty} \nabla f(x + \alpha d)' d, \qquad \forall\, x, d \in \Re^n$$

if f is differentiable.

- Calculus of recession functions:

$$r_{f_1 + \cdots + f_m}(d) = r_{f_1}(d) + \cdots + r_{f_m}(d),$$

$$r_{\sup_{i \in I} f_i}(d) = \sup_{i \in I} r_{f_i}(d)$$

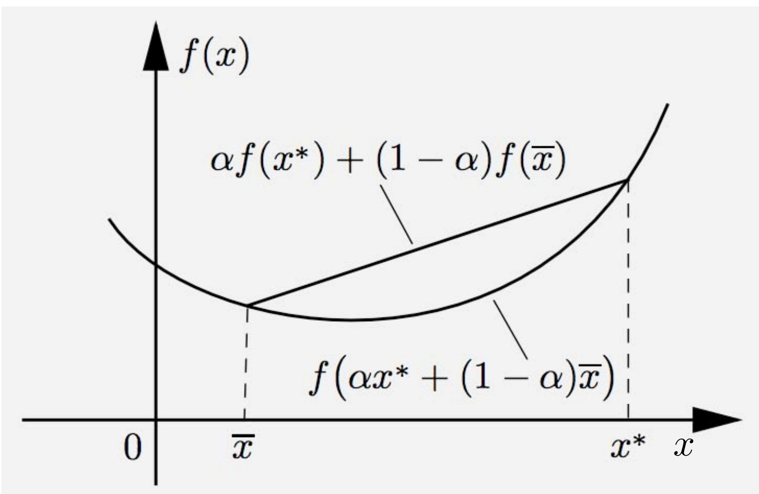
LOCAL AND GLOBAL MINIMA

- Consider minimizing $f : \Re^n \mapsto (-\infty, \infty]$ over a set $X \subset \Re^n$
- x is **feasible** if $x \in X \cap \mathrm{dom}(f)$
- x^* is a (global) **minimum** of f over X if x^* is feasible and $f(x^*) = \inf_{x \in X} f(x)$
- x^* is a **local minimum** of f over X if x^* is a minimum of f over a set $X \cap \{x \mid \|x - x^*\| \leq \epsilon\}$

Proposition: If X is convex and f is convex, then:

(a) A local minimum of f over X is also a global minimum of f over X.

(b) If f is strictly convex, then there exists at most one global minimum of f over X.



EXISTENCE OF OPTIMAL SOLUTIONS

- The set of minima of a proper $f : \Re^n \mapsto (-\infty, \infty]$ is the intersection of its nonempty level sets.

- The set of minima of f is nonempty and compact if the level sets of f are compact.

- **(An Extension of the) Weierstrass' Theorem:** The set of minima of f over X is nonempty and compact if X is closed, f is lower semicontinuous over X, and one of the following conditions holds:

  (1) X is bounded.

  (2) Some set $\{x \in X \mid f(x) \leq \gamma\}$ is nonempty and bounded.

  (3) For every sequence $\{x_k\} \subset X$ s. t. $\|x_k\| \to \infty$, we have $\lim_{k\to\infty} f(x_k) = \infty$. (Coercivity property).

Proof: In all cases the level sets of $f \cap X$ are compact. **Q.E.D.**

EXISTENCE OF SOLUTIONS - CONVEX CASE

- **Weierstrass' Theorem specialized to convex functions:** Let X be a closed convex subset of $\Re^n$, and let $f : \Re^n \mapsto (-\infty, \infty]$ be closed convex with $X \cap \text{dom}(f) \neq \emptyset$. The set of minima of f over X is nonempty and compact if and only if X and f have no common nonzero direction of recession.

Proof: Let $f^* = \inf_{x \in X} f(x)$ and note that $f^* < \infty$ since $X \cap \text{dom}(f) \neq \emptyset$. Let $\{\gamma_k\}$ be a scalar sequence with $\gamma_k \downarrow f^*$, and consider the sets

$$V_k = \big\{x \mid f(x) \leq \gamma_k\big\}.$$

Then the set of minima of f over X is

$$X^* = \cap_{k=1}^{\infty} (X \cap V_k).$$

The sets $X \cap V_k$ are nonempty and have $R_X \cap R_f$ as their common recession cone, which is also the recession cone of X^*, when $X^* \neq \emptyset$. It follows that X^* is nonempty and compact if and only if $R_X \cap R_f = \{0\}$. **Q.E.D.**

EXISTENCE OF SOLUTION, SUM OF FNS

• Let $f_i : \Re^n \mapsto (-\infty, \infty]$, $i = 1, \ldots, m$, be closed proper convex functions such that the function

$$f = f_1 + \cdots + f_m$$

is proper. Assume that a single function f_i satisfies $r_{f_i}(d) = \infty$ for all $d \neq 0$. Then the set of minima of f is nonempty and compact.

• **Proof:** We have $r_f(d) = \infty$ for all $d \neq 0$ since $r_f(d) = \sum_{i=1}^m r_{f_i}(d)$. Hence f has no nonzero directions of recession. **Q.E.D.**

• True also for $f = \max\{f_1, \ldots, f_m\}$.

• **Example of application:** If one of the f_i is positive definite quadratic, the set of minima of the sum f is nonempty and compact.

• Also f has a unique minimum because the positive definite quadratic is strictly convex, which makes f strictly convex.

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