MASSACHUSETTS INSTITUTE OF TECHNOLOGY
2012 Spring 6.253 Midterm Exam

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Problem 1. (60 points) In the following, X is a nonempty convex subset of $\Re^n$, A is a matrix of appropriate dimension, b is a vector of appropriate dimension, and $f : \Re^n \mapsto (-\infty, \infty]$ is a convex proper function. State which of the following statements are true and which are false. You don't have to justify your answers.

1. If the epigraph of f is closed, f is continuous.
2. If the epigraph of f is closed, $\text{dom}(f)$ is closed.
3. The relative interior of X is equal to its interior.
4. The recession cone of X is equal to the recession cone of its relative interior.
5. There exists a hyperplane that separates X and $-X$.
6. Let f be the two-dimensional function $f(x_1, x_2) = (x_1 + x_2)^2$. Then f is coercive.
7. If f is closed and $\text{dom}(f)$ is compact then its conjugate is real-valued.
8. Suppose that the problem of minimizing f over $x \in X$ and $Ax = b$ has finite optimal value and X is open. Then there is no duality gap.
9. Suppose f is the sum of a real-valued function and the indicator function of X. Then at each $x \in X$ there is at least one subgradient of f.
10. Suppose $f(x) = b'x$ and x^* minimizes f over X. Then the normal cone of X at x^* contains $-b$.

Solution.

1. False. The given conditions only ensure that f is lower semicontinuous.
2. False. Consider the function $f : \Re^+ \mapsto \Re^+$ defined by $f(x) = 1/x$.
3. False. Consider the interval $[-1, 1]$ on an axis of $\Re^2$, which has empty interior but nonempty relative interior $(-1, 1)$.
4. False. Consider $X = \{x_1 > 0, x_2 \geq 0\} \cup \{(0, 0\}$. Then the recession cone of X is X itself, while the recession cone of the relative interior of X is the positive orthant.
5. False. If 0 is an interior point of X, then X and $-X$ cannot be separated.
6. False. Along the line $x_1 + x_2 = 0$, even if $\sqrt{x_1^2 + x_2^2} \to \infty$, we have $f(x_1, x_2) = 0$.
7. True.
8. True. By the strong duality theorem.
9. True. Subgradients of the real-valued function and the indicator function of X exist at all $x \in X$, and the relative interiors of their domains intersect. Use Prop. 5.4.6.
10. True.

Problem 2. (40 points) Consider the problem

$$\min \quad x^2 + y^2$$

$$\text{subject to } a - x - y \le 0, \quad x, y \in \{0, 1\}.$$

(a) Sketch the set of constraint-cost pairs:

$$\big\{(a - x - y, x^2 + y^2) \mid x, y \in \{0, 1\}\big\},$$

and the perturbation function

$$p(u) = \min_{a - x - y \le u, \quad x, y \in \{0,1\}} x^2 + y^2.$$

Is p lower semicontinuous?

(b) Consider the MC/MC framework with M being the epigraph of p. What are the values of a for which the problem is feasible and at the same time there is a duality gap? What are the values of a for which the problem is feasible and there is no duality gap? What are the values of a for which the problem is feasible and has a unique dual solution?

(c) Formulate the max crossing problem for one of the values of a for which the problem is feasible and there is no duality gap, and find the set of primal and dual optimal solutions.

(d) Replace the constraint $a - x - y \le 0$ with a strict inequality $a - x - y < 0$. Answer the questions in parts (a) and (b) again.

Solution. (a) To be added. The perturbation function is

$$p(u) = \begin{cases} 0 & \text{if } u \ge a, \\ 1 & \text{if } a > u \ge a - 1, \\ 2 & \text{if } a - 1 > u \ge a - 2, \\ \infty & \text{if } a - 2 > u, \end{cases}$$

We can show that p is lower semicontinuous by verifying the definition.

(b) For the problem to be feasible, we must have $a \le 2$; and for there is a duality gap, from the MC/MC we see that $a > 0$, $a \ne 1$ and $a \ne 2$. To sum up, we have $a \in (0, 1) \cup (1, 2)$.

For the problem to be feasible with no duality gap, we must have $a \in (-\infty, 0] \cup \{1, 2\}$.

For the problem to be feasible with a unique dual solution, we have $a \in (-\infty, 0) \cup (0, 2)$.

(c) Let $a = 1$. The optimal solution is $(x^*, y^*) = (0, 1)$ or $(x^*, y^*) = (1, 0)$ and the optimal value is $f^* = 1$. The max crossing problem is

$$\max_{\mu \ge 0} \inf_{u \in \Re} \big\{p(u) + \mu' u\big\},$$

and the solution is $q^* = 1$ and $\mu^* = 1$.

(d) The function p is no longer semi-continuous. For the problem to be feasible with a duality gap, we must have $a \in [0, 2)$. For the problem to be feasible with no duality gap, we must have $a \in (-\infty, 0)$. For the problem to be feasible and has a unique dual solution, we have $a \in (-\infty, 0) \cup (0, 2)$.

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