6.253: Convex Analysis and Optimization
Homework 1

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Problem 1

(a) Let C be a nonempty subset of $\mathbf{R}^n$, and let λ_1 and λ_2 be positive scalars. Show that if C is convex, then $(\lambda_1 + \lambda_2)C = \lambda_1 C + \lambda_2 C$. Show by example that this need not be true when C is not convex.
(b) Show that the intersection $\cap_{i \in I} C_i$ of a collection $\{C_i \mid i \in I\}$ of cones is a cone.
(c) Show that the image and the inverse image of a cone under a linear transformation is a cone.
(d) Show that the vector sum $C_1 + C_2$ of two cones C_1 and C_2 is a cone.
(e) Show that a subset C is a convex cone if and only if it is closed under addition and positive scalar multiplication, i.e., $C + C \subset C$, and $\gamma C \subset C$ for all $\gamma > 0$.

Problem 2

Let C be a nonempty convex subset of $\mathbf{R}^n$. Let also $f = (f_1, \ldots, f_m)$, where $f_i : C \mapsto \Re$, $i = 1, \ldots, m$, are convex functions, and let $g : \mathbf{R}^m \mapsto \mathbf{R}$ be a function that is convex and monotonically nondecreasing over a convex set that contains the set $\{f(x) \mid x \in C\}$, in the sense that for all u_1, u_2 in this set such that $u_1 \leq u_2$, we have $g(u_1) \leq g(u_2)$. Show that the function h defined by $h(x) = g(f(x))$ is convex over C. If in addition, $m = 1$, g is monotonically increasing and f is strictly convex, then h is strictly convex.

Problem 3

Show that the following functions from $\mathbf{R}^n$ to $(-\infty, \infty]$ are convex:
(a) $f_1(x) = \ln(e^{x_1} + \cdots + e^{x_n})$.
(b) $f_2(x) = \|x\|^p$ with $p \geq 1$.
(c) $f_3(x) = e^{\beta x' A x}$, where A is a positive semidefinite symmetric $n \times n$ matrix and β is a positive scalar.
(d) $f_4(x) = f(Ax + b)$, where $f : \mathbf{R}^m \mapsto \mathbf{R}$ is a convex function, A is an $m \times n$ matrix, and b is a vector in $\mathbf{R}^m$.

Problem 4

Let X be a nonempty bounded subset of $\mathbf{R}^n$. Show that

$$cl(conv(X)) = conv(cl(X)).$$

In particular, if X is compact, then $\text{conv}(X)$ is compact.

Problem 5

Construct an example of a point in a nonconvex set X that has the prolongation property, but is not a relative interior point of X.