6.252 NONLINEAR PROGRAMMING

LECTURE 21: DUAL COMPUTATIONAL METHODS

LECTURE OUTLINE

- Dual Methods
- Nondifferentiable Optimization

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- Consider the primal problem

minimize $f(x)$

subject to $x \in X, \qquad g_j(x) \leq 0, \qquad j = 1, \ldots, r,$

assuming $-\infty < f^* < \infty$.

- Dual problem: Maximize

$$q(\mu) = \inf_{x \in X} L(x, \mu) = \inf_{x \in X} \{f(x) + \mu' g(x)\}$$

subject to $\mu \geq 0$.

PROS AND CONS FOR SOLVING THE DUAL

- The dual is concave.
- The dual may have smaller dimension and/or simpler constraints.
- If there is no duality gap and the dual is solved exactly for a Lagrange multiplier μ^*, all optimal primal solutions can be obtained by minimizing the Lagrangian $L(x, \mu^*)$ over $x \in X$.
- Even if there is a duality gap, $q(\mu)$ is a lower bound to the optimal primal value for every $\mu \geq 0$.
- Evaluating $q(\mu)$ requires minimization of $L(x, \mu)$ over $x \in X$.
- The dual function is often nondifferentiable.
- Even if we find an optimal dual solution μ^*, it may be difficult to obtain a primal optimal solution.

STRUCTURE

- Separability: Classical duality structure (Lagrangian relaxation).
- Partitioning: The problem

$$\begin{aligned} &\text{minimize} \quad F(x) + G(y) \\ &\text{subject to} \quad Ax + By = c, \quad x \in X, \quad y \in Y \end{aligned}$$

can be written as

$$\begin{aligned} &\text{minimize} \quad F(x) + \inf_{By = c - Ax,\, y \in Y} G(y) \\ &\text{subject to} \quad x \in X. \end{aligned}$$

With no duality gap, this problem is written as

$$\begin{aligned} &\text{minimize} \quad F(x) + Q(Ax) \\ &\text{subject to} \quad x \in X, \end{aligned}$$

where

$$Q(Ax) = \max_{\lambda} q(\lambda, Ax)$$

$$q(\lambda, Ax) = \inf_{y \in Y} \Big\{ G(y) + \lambda'(Ax + By - c) \Big\}$$

DUAL DERIVATIVES

- Let

$$x_\mu = \arg\min_{x\in X} L(x,\mu) = \arg\min_{x\in X}\Big\{f(x) + \mu' g(x)\Big\}.$$

Then for all $\overline{\mu} \in \Re^r$,

$$\begin{aligned} q(\tilde{\mu}) &= \inf_{x\in X}\Big\{f(x) + \tilde{\mu}' g(x)\Big\} \\ &\le f(x_\mu) + \tilde{\mu}' g(x_\mu) \\ &= f(x_\mu) + \mu' g(x_\mu) + (\tilde{\mu} - \mu)' g(x_\mu) \\ &= q(\mu) + (\tilde{\mu} - \mu)' g(x_\mu). \end{aligned}$$

- Thus $g(x_\mu)$ *is a subgradient of* q *at* μ.

- Proposition: Let X be compact, and let f and g be continuous over X. Assume also that for every μ, $L(x,\mu)$ is minimized over $x \in X$ at a unique point x_μ. Then, q is everywhere continuously differentiable and

$$\nabla q(\mu) = g(x_\mu), \qquad \forall\ \mu \in \Re^r.$$

NONDIFFERENTIABLE DUAL

- If there exists a duality gap, the dual function is nondifferentiable at every dual optimal solution.
- Important nondifferentiable case: When q is polyhedral, that is,

$$q(\mu) = \min_{i \in I} \left\{ a_i' \mu + b_i \right\},$$

where I is a finite index set, and $a_i \in \Re^r$ and b_i are given (arises when X is a discrete set, as in integer programming).

- Proposition: Let q be polyhedral as above, and let I_μ be the set of indices attaining the minimum

$$I_\mu = \left\{ i \in I \mid a_i' \mu + b_i = q(\mu) \right\}.$$

The set of all subgradients of q at μ is

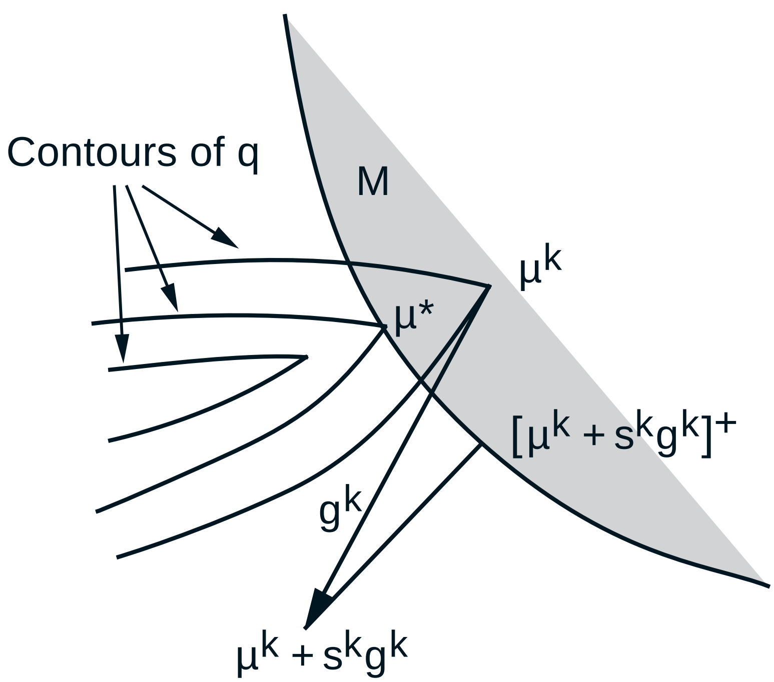
$$\partial q(\mu) = \left\{ g \;\middle|\; g = \sum_{i \in I_\mu} \xi_i a_i,\, \xi_i \geq 0,\, \sum_{i \in I_\mu} \xi_i = 1 \right\}.$$

NONDIFFERENTIABLE OPTIMIZATION

- Consider maximization of $q(\mu)$ over $M = \{| \ \mu \geq 0, \ q(\mu) > -\infty\}$
- Subgradient method:

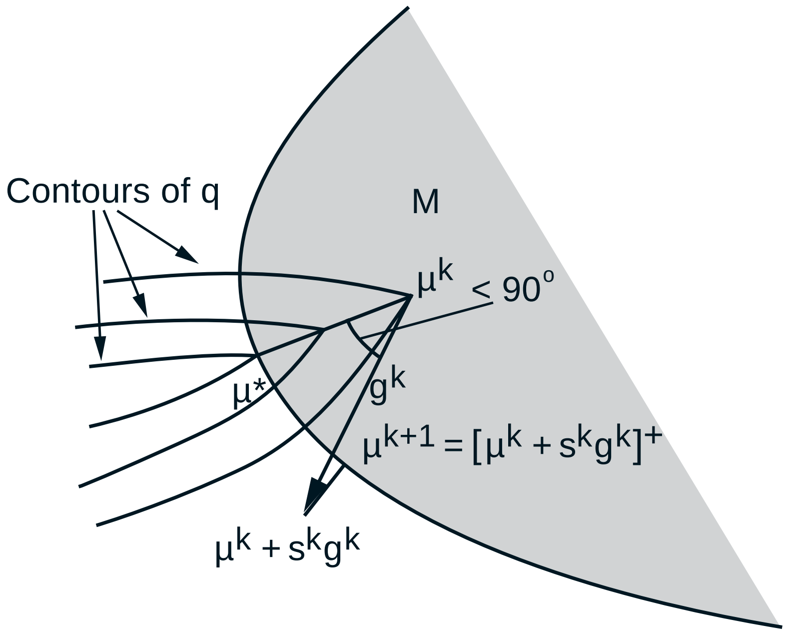
$$\mu^{k+1} = \left[\mu^k + s^k g^k\right]^+,$$

where g^k is the subgradient $g(x_{\mu^k})$, $[\cdot]^+$ denotes projection on the closed convex set M, and s^k is a positive scalar stepsize.



KEY SUBGRADIENT METHOD PROPERTY

- For a small stepsize it reduces the Euclidean distance to the optimum.



- Proposition: For any dual optimal solution μ^*, we have

$$\|\mu^{k+1} - \mu^*\| < \|\mu^k - \mu^*\|,$$

for all stepsizes s^k such that

$$0 < s^k < \frac{2\Big(q(\mu^*) - q(\mu^k)\Big)}{\|g^k\|^2}.$$

STEPSIZE RULES

- Diminishing stepsize is one possibility.
- More common method:

$$s^k = \frac{\alpha^k \big(q^k - q(\mu^k)\big)}{\|g^k\|^2},$$

where $q^k \approx q^*$ and

$$0 < \alpha^k < 2.$$

- Some possibilities:
  - q^k is the best known upper bound to q^*; $\alpha^0 = 1$ and α^k decreased by a certain factor every few iterations.
  - $\alpha^k = 1$ for all k and

$$q^k = \big(1 + \beta(k)\big)\hat{q}^k,$$

where $\hat{q}^k = \max_{0 \le i \le k} q(\mu^i)$, and $\beta(k) > 0$ is adjusted depending on algorithmic progress of the algorithm.