6.252 NONLINEAR PROGRAMMING

LECTURE 20: STRONG DUALITY

LECTURE OUTLINE

- Strong Duality Theorem
- Linear equality constraints. Fenchel Duality.

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- Consider the problem

$$\begin{aligned} &\text{minimize } f(x) \\ &\text{subject to } x \in X, \qquad g_j(x) \le 0, \qquad j = 1, \dots, r, \end{aligned}$$

assuming $-\infty < f^* < \infty$.

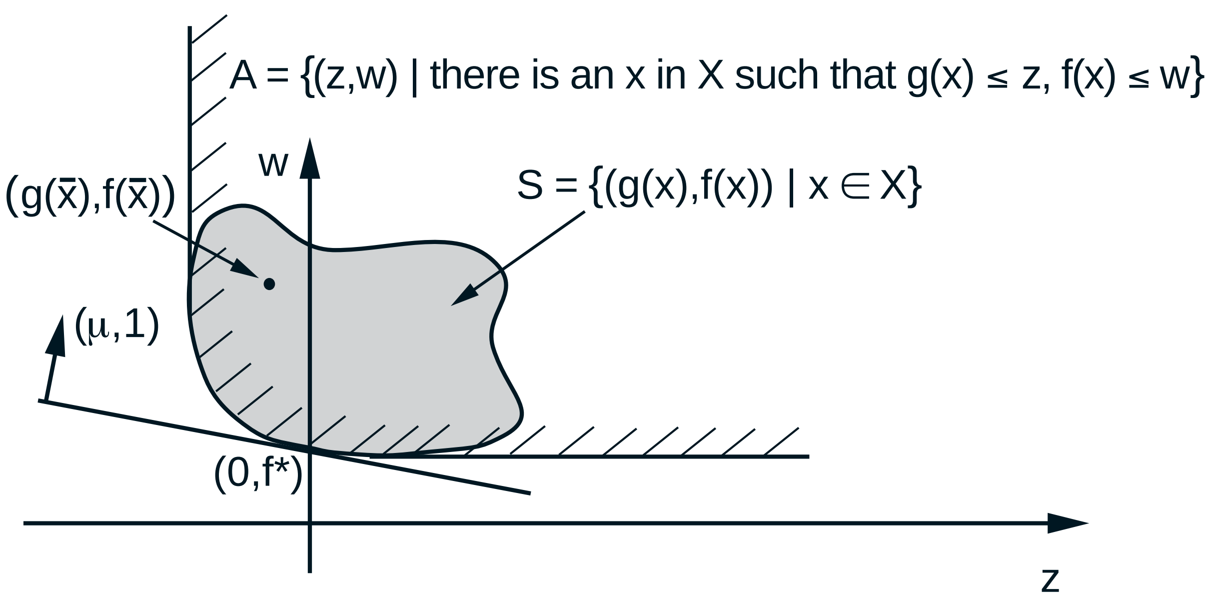
- μ^* is a Lagrange multiplier if $\mu^* \ge 0$ and $f^* = \inf_{x \in X} L(x, \mu^*)$.
- Dual problem: Maximize $q(\mu) = \inf_{x \in X} L(x, \mu)$ subject to $\mu \ge 0$.

DUALITY THEOREM FOR INEQUALITIES

- Assume that X is convex and the functions $f : \Re^n \mapsto \Re$, $g_j : \Re^n \mapsto \Re$ are convex over X. Furthermore, the optimal value f^* is finite and there exists a vector $\bar{x} \in X$ such that

$$g_j(\bar{x}) < 0, \qquad \forall\ j = 1, \ldots, r.$$

- Strong Duality Theorem: There exists at least one Lagrange multiplier and there is no duality gap.



PROOF OUTLINE

- Show that A is convex. [Consider vectors $(z, w) \in A$ and $(\tilde{z}, \tilde{w}) \in A$, and show that their convex combinations lie in A.]
- Observe that $(0, f^*)$ is not an interior point of A.
- Hence, there is hyperplane passing through $(0, f^*)$ and containing A in one of the two corresponding halfspaces; i.e., a $(\mu, \beta) \neq (0, 0)$ with

$$\beta f^* \leq \beta w + \mu' z, \qquad \forall\ (z, w) \in A.$$

This implies that $\beta \geq 0$, and $\mu_j \geq 0$ for all j.

- Prove that hyperplane is nonvertical, i.e., $\beta > 0$.
- Normalize ($\beta = 1$), take the infimum over $x \in X$, and use the fact $\mu \geq 0$, to obtain

$$f^* \leq \inf_{x \in X} \big\{ f(x) + \mu' g(x) \big\} = q(\mu) \leq \sup_{\mu \geq 0} q(\mu) = q^*.$$

Using the weak duality theorem, μ is a Lagrange multiplier and there is no duality gap.

LINEAR EQUALITY CONSTRAINTS

- Suppose we have the additional constraints

$$e_i'x - d_i = 0, \qquad i = 1, \ldots, m$$

- We need the notion of the *affine hull* of a convex set X [denoted $aff(X)$]. This is the intersection of all hyperplanes containing X.

- The *relative interior* of X, denoted $ri(X)$, is the set of all $x \in X$ s.t. there exists $\epsilon > 0$ with

$$\Big\{z \mid \|z - x\| < \epsilon,\ z \in aff(X)\Big\} \subset X,$$

that is, $ri(X)$ is the interior of X relative to $aff(X)$.

- Every nonempty convex set has a nonempty relative interior.

DUALITY THEOREM FOR EQUALITIES

- Assumptions:
  - The set X is convex and the functions f, g_j are convex over X.
  - The optimal value f^* is finite and there exists a vector $\bar{x} \in ri(X)$ such that

$$g_j(\bar{x}) < 0, \qquad j = 1, \ldots, r,$$

$$e_i'\bar{x} - d_i = 0, \qquad i = 1, \ldots, m.$$

- Under the preceding assumptions there exists at least one Lagrange multiplier and there is no duality gap.

COUNTEREXAMPLE

- Consider

minimize $f(x) = x_1$

subject to $x_2 = 0, \quad x \in X = \left\{(x_1, x_2) \mid x_1^2 \leq x_2\right\}.$

- The optimal solution is $x^* = (0,0)$ and $f^* = 0$.
- The dual function is given by

$$q(\lambda) = \inf_{x_1^2 \leq x_2} \{x_1 + \lambda x_2\} = \begin{cases} -\frac{1}{4\lambda}, & \text{if } \lambda > 0, \\ -\infty, & \text{if } \lambda \leq 0. \end{cases}$$

- No dual optimal solution and therefore there is no Lagrange multiplier. (Even though there is no duality gap.)
- Assumptions are violated (the feasible set and the relative interior of X have no common point).

FENCHEL DUALITY FRAMEWORK

- Consider the problem

$$\begin{aligned} &\text{minimize} \quad f_1(x) - f_2(x) \\ &\text{subject to} \quad x \in X_1 \cap X_2, \end{aligned}$$

where f_1 and f_2 are real-valued functions on $\Re^n$, and X_1 and X_2 are subsets of $\Re^n$.

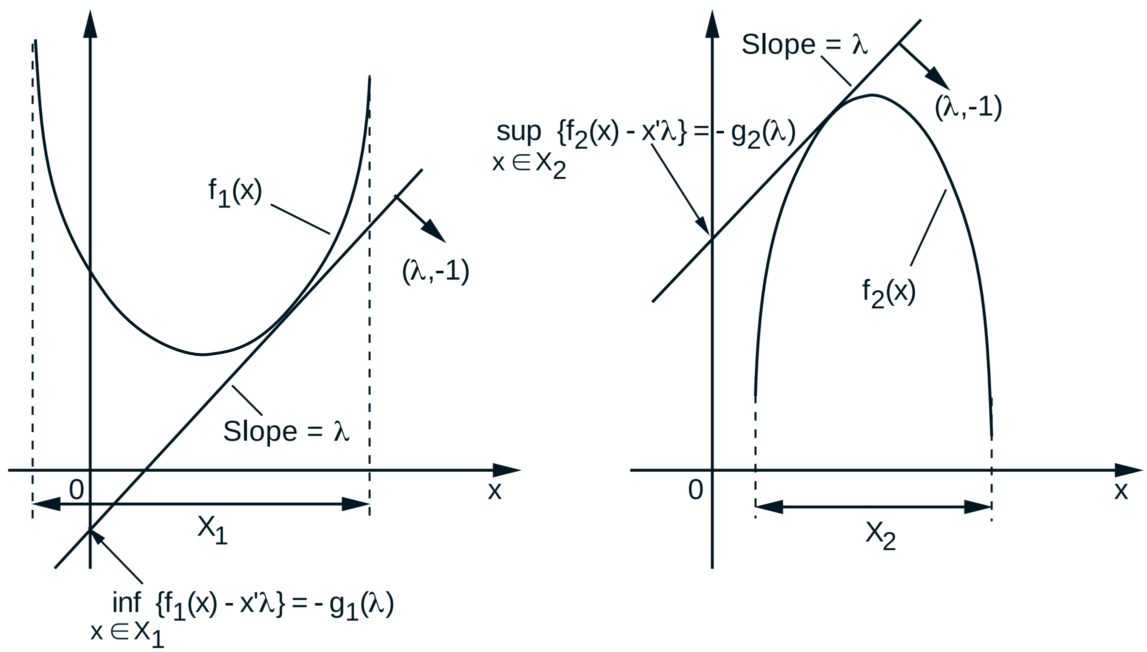
- Assume that $-\infty < f^* < \infty$.
- Convert problem to

$$\begin{aligned} &\text{minimize} \quad f_1(y) - f_2(z) \\ &\text{subject to} \quad z = y, \qquad y \in X_1, \qquad z \in X_2, \end{aligned}$$

and dualize the constraint $z = y$.

$$\begin{aligned} q(\lambda) &= \inf_{y \in X_1,\, z \in X_2} \left\{ f_1(y) - f_2(z) + (z - y)'\lambda \right\} \\ &= \inf_{z \in X_2} \left\{ z'\lambda - f_2(z) \right\} - \sup_{y \in X_1} \left\{ y'\lambda - f_1(y) \right\} \\ &= g_2(\lambda) - g_1(\lambda) \end{aligned}$$

DUALITY THEOREM



- Assume that
  - X_1 and X_2 are convex
  - f_1 and f_2 are convex and concave over X_1 and X_2, respectively
  - The relative interiors of X_1 and X_2 intersect
- The duality theorem for equalities applies and shows that

$$f^* = \max_{\lambda \in \Re^n} \Big\{ g_2(\lambda) - g_1(\lambda) \Big\}$$

and that the maximum above is attained.