6.252 NONLINEAR PROGRAMMING

LECTURE 18: DUALITY THEORY

LECTURE OUTLINE

- Geometrical Framework for Duality
- Lagrange Multipliers
- The Dual Problem
- Properties of the Dual Function

- Consider the problem

$$\text{minimize } f(x)$$
$$\text{subject to } x \in X, \qquad g_j(x) \leq 0, \qquad j = 1, \ldots, r,$$

assuming $-\infty < f^* < \infty$.

- We assume that the problem is feasible and the cost is bounded from below,

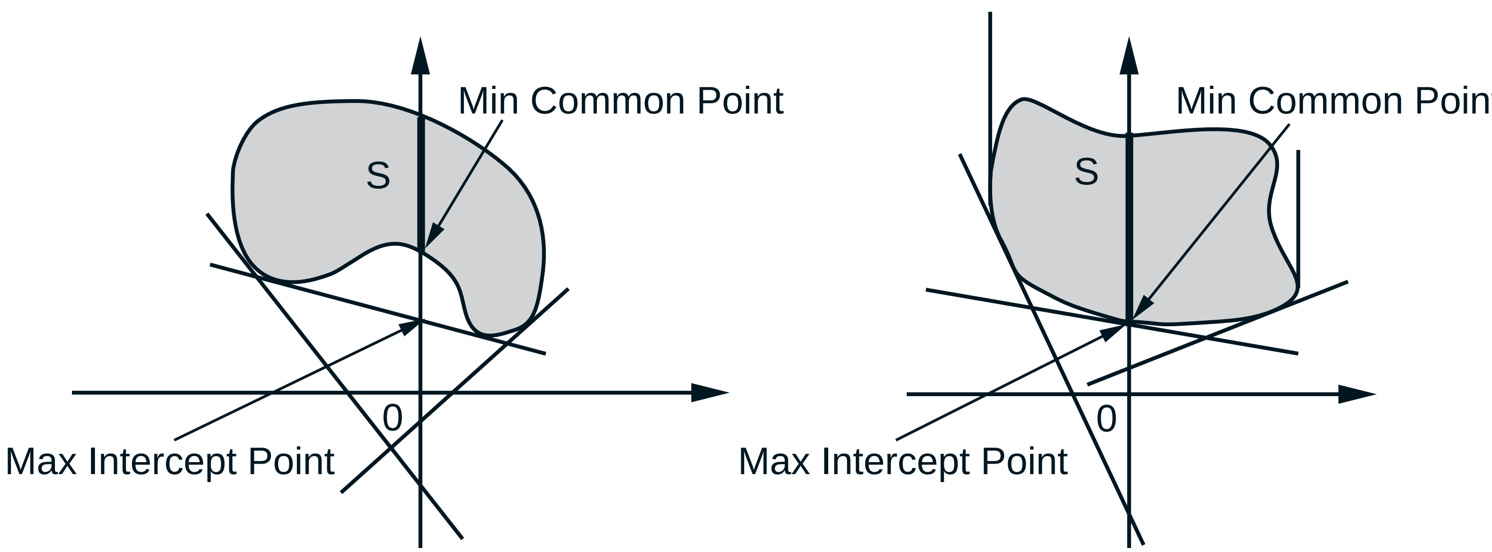
$$-\infty < f^* = \inf_{\substack{x \in X \\ g_j(x) \leq 0,\, j=1,\ldots,r}} f(x) < \infty$$

MIN COMMON POINT/MAX INTERCEPT POINT

- Let S be a subset of $\Re^n$:

- $\mathit{Min\ Common\ Point\ Problem}$: Among all points that are common to both S and the nth axis,find the one whose nth component is minimum.

- $\mathit{Max\ Intercept\ Point\ Problem}$: Among all hyperplanes that intersect the nth axis and support the set S from "below", find the hyperplane for which point of intercept with the nth axis is maximum.



(a) (b)

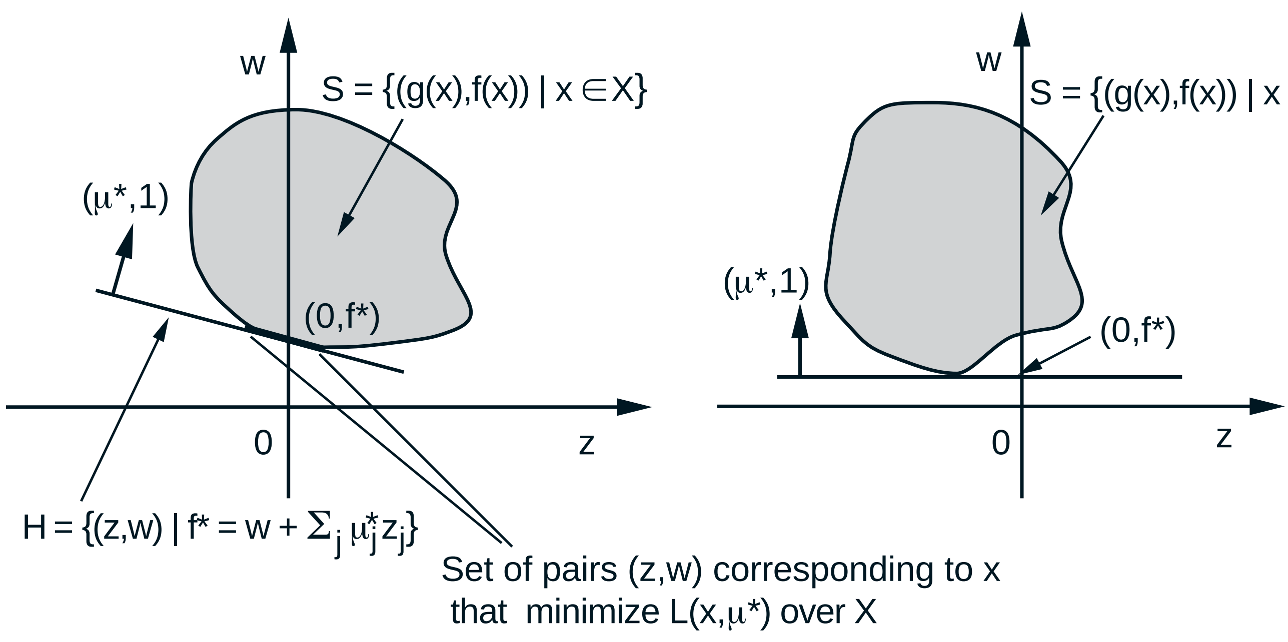
GEOMETRICAL DEFINITION OF A L-MULTIPLIER

- A vector $\mu^* = (\mu_1^*, \ldots, \mu_r^*)$ is said to be a *Lagrange multiplier* for the primal problem if

$$\mu_j^* \geq 0, \qquad j = 1, \ldots, r,$$

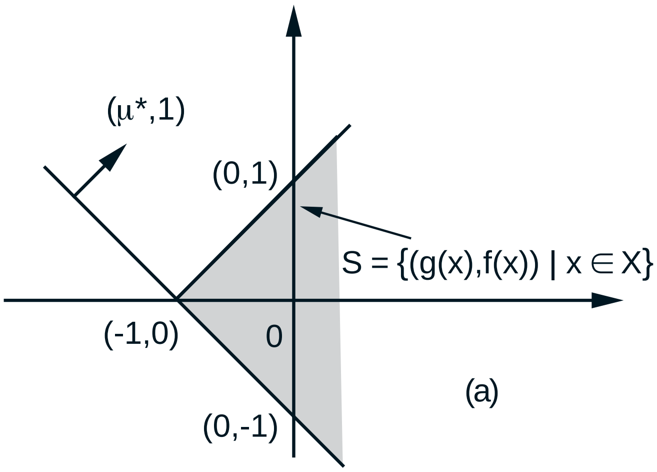
and

$$f^* = \inf_{x \in X} L(x, \mu^*).$$

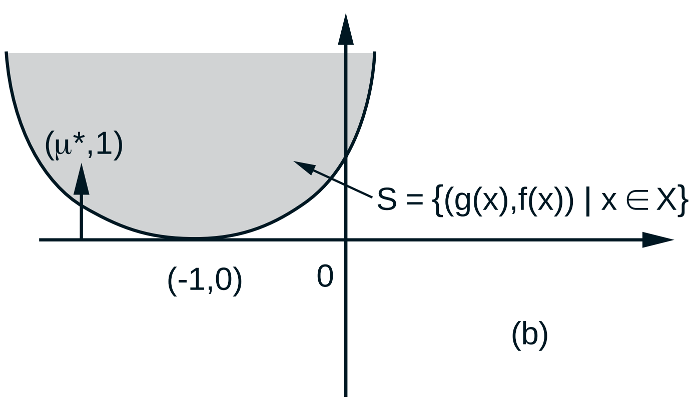


(a) (b)

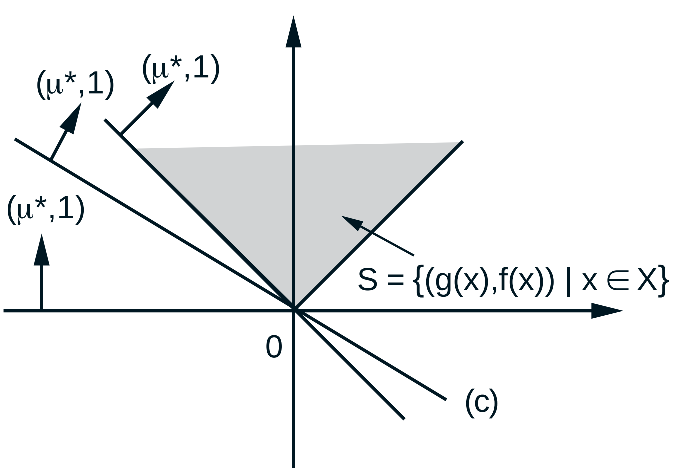
EXAMPLES: A L-MULTIPLIER EXISTS



$$\min\ f(x) = x_1 - x_2$$
$$\text{s.t. } g(x) = x_1 + x_2 - 1 \leq 0$$
$$x \in X = \{(x_1, x_2) \mid x_1 \geq 0,\ x_2 \geq 0\}$$

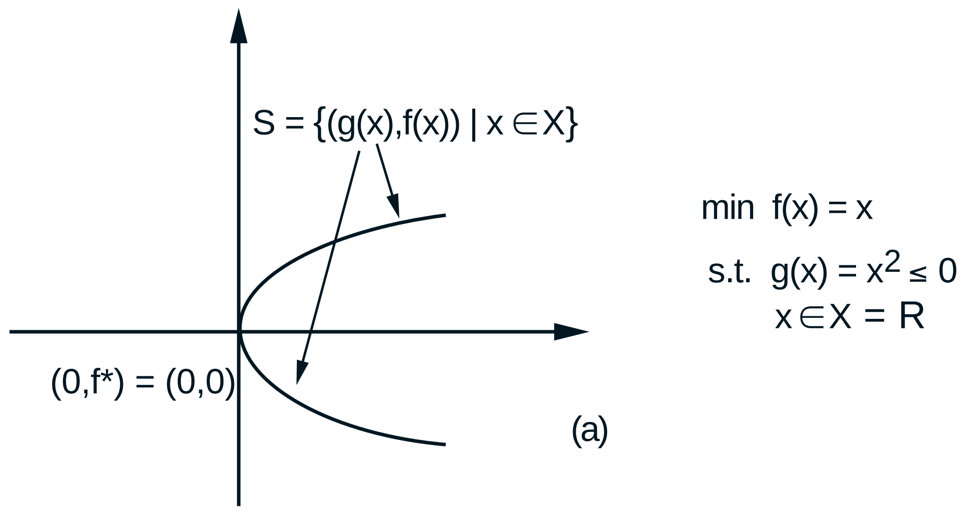


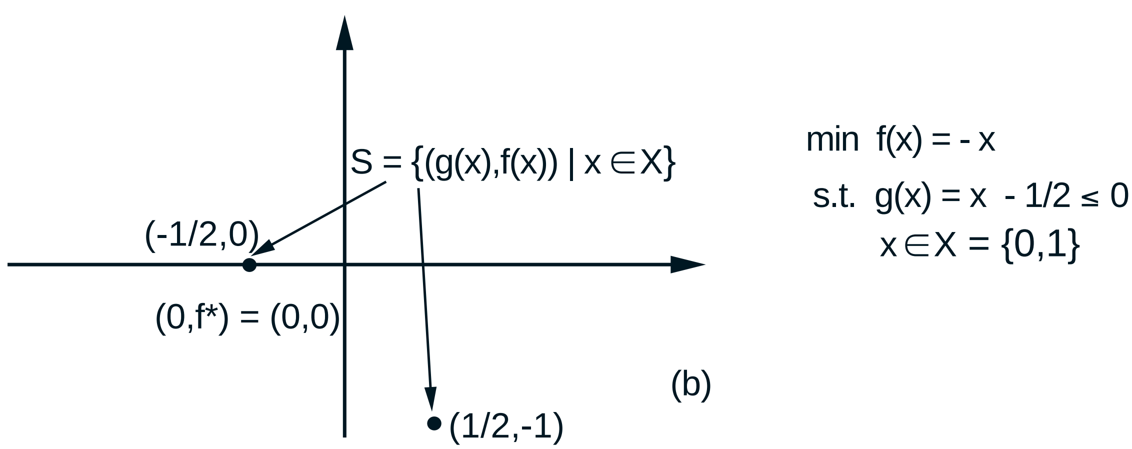
$$\min\ f(x) = (1/2)\,(x_1^2 + x_2^2)$$
$$\text{s.t. } g(x) = x_1 - 1 \leq 0$$
$$x \in X = R^2$$



$$\min\ f(x) = |x_1| + x_2$$
$$\text{s.t. } g(x) = x_1 \leq 0$$
$$x \in X = \{(x_1, x_2) \mid x_2 \geq 0\}$$

EXAMPLES: A L-MULTIPLIER DOESN'T EXIST





- Proposition: Let μ^* be a Lagrange multiplier. Then x^* is a global minimum of the primal problem if and only if x^* is feasible and

$$x^* = \arg\min_{x\in X} L(x,\mu^*), \qquad \mu_j^* g_j(x^*) = 0, \qquad j = 1,\ldots,r$$

THE DUAL FUNCTION AND THE DUAL PROBLEM

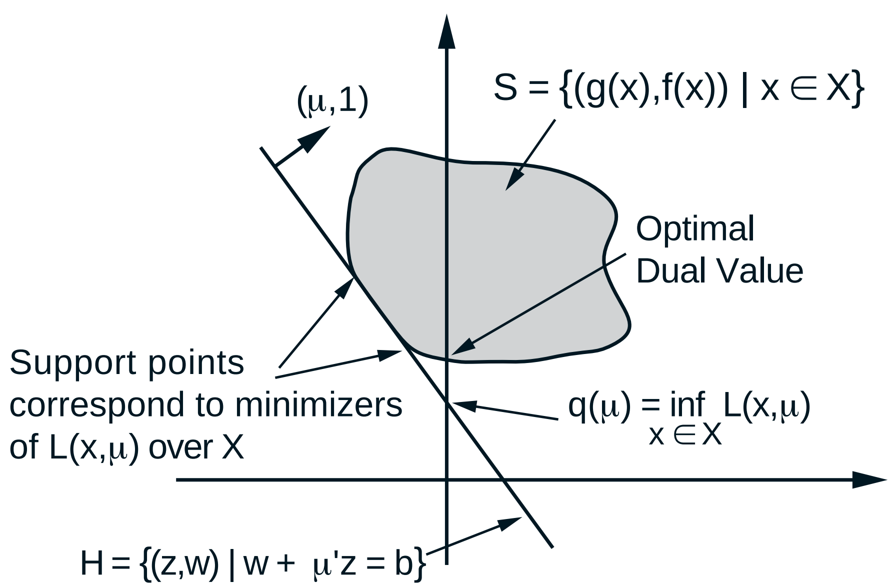
- The *dual problem* is

$$\text{maximize } \; q(\mu)$$
$$\text{subject to } \; \mu \geq 0,$$

where q is the dual function

$$q(\mu) = \inf_{x \in X} L(x, \mu), \qquad \forall \; \mu \in \Re^r.$$

- Question: How does the optimal dual value $q^* = \sup_{\mu \geq 0} q(\mu)$ relate to f^*?



WEAK DUALITY

- The $domain$ of q is

$$D_q = \Big\{\mu \mid q(\mu) > -\infty\Big\}.$$

- Proposition: The domain D_q is a convex set and q is concave over D_q.
- Proposition: (Weak Duality Theorem) We have

$$q^* \le f^*.$$

Proof: For all $\mu \ge 0$, and $x \in X$ with $g(x) \le 0$, we have

$$q(\mu) = \inf_{z \in X} L(z, \mu) \le f(x) + \sum_{j=1}^{r} \mu_j g_j(x) \le f(x),$$

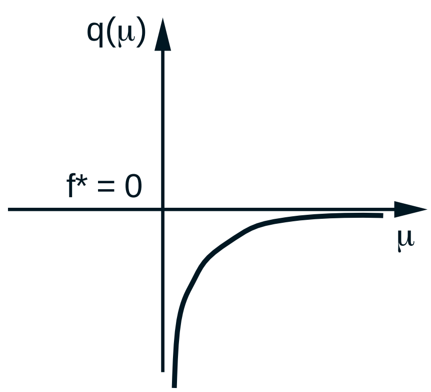
so

$$q^* = \sup_{\mu \ge 0} q(\mu) \le \inf_{x \in X,\, g(x) \le 0} f(x) = f^*.$$

DUAL OPTIMAL SOLUTIONS AND L-MULTIPLIERS

- Proposition: (a) If $q^* = f^*$, the set of Lagrange multipliers is equal to the set of optimal dual solutions. (b) If $q^* < f^*$, the set of Lagrange multipliers is empty.

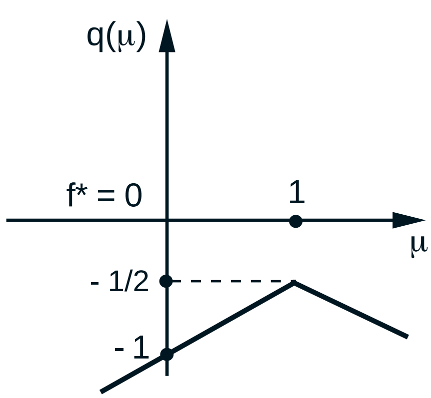
Proof: By definition, a vector $\mu^* \geq 0$ is a Lagrange multiplier if and only if $f^* = q(\mu^*) \leq q^*$, which by the weak duality theorem, holds if and only if there is no duality gap and μ^* is a dual optimal solution. **Q.E.D.**



$$\min\ f(x) = x$$
$$\text{s.t. } g(x) = x^2 \leq 0$$
$$x \in X = R$$

$$q(\mu) = \min_{x \in R} \{x + \mu x^2\} = \begin{cases} -1/(4\mu) & \text{if } \mu > 0 \\ -\infty & \text{if } \mu \leq 0 \end{cases}$$

(a)



$$\min\ f(x) = -x$$
$$\text{s.t. } g(x) = x - 1/2 \leq 0$$
$$x \in X = \{0,1\}$$

$$q(\mu) = \min_{x \in \{0,1\}} \{-x + \mu(x - 1/2)\} = \min\{-\mu/2, \mu/2 - 1\}$$

(b)