6.252 NONLINEAR PROGRAMMING

LECTURE 14: INTRODUCTION TO DUALITY

LECTURE OUTLINE

- Convex Cost/Linear Constraints
- Duality Theorem
- Linear Programming Duality
- Quadratic Programming Duality

Linear inequality constrained problem

$$\begin{aligned} &\text{minimize } f(x) \\ &\text{subject to } a_j' x \le b_j, \qquad j = 1, \ldots, r, \end{aligned}$$

where f is convex and continuously differentiable over $\Re^n$.

LAGRANGE MULTIPLIER RESULT

Let $J \subset \{1, \ldots, r\}$. Then x^* is a global min if and only if x^* is feasible and there exist $\mu_j^* \geq 0$, $j \in J$, such that $\mu_j^* = 0$ for all $j \in J \notin A(x^*)$, and

$$x^* = \arg \min_{\substack{a_j' x \leq b_j \\ j \notin J}} \left\{ f(x) + \sum_{j \in J} \mu_j^* (a_j' x - b_j) \right\}.$$

Proof: Assume x^* is global min. Then there exist $\mu_j^* \geq 0$, such that $\mu_j^*(a_j' x^* - b_j) = 0$ for all j and $\nabla f(x^*) + \sum_{j=1}^r \mu_j^* a_j = 0$, implying

$$x^* = \arg \min_{x \in \Re^n} \left\{ f(x) + \sum_{j=1}^{r} \mu_j^* (a_j' x - b_j) \right\}.$$

Since $\mu_j^*(a_j' x^* - b_j) = 0$ for all j,

$$f(x^*) = \min_{x \in \Re^n} \left\{ f(x) + \sum_{j=1}^{r} \mu_j^* (a_j' x - b_j) \right\}.$$

Since $\mu_j^*(a_j' x - b_j) \leq 0$ if $a_j' x - b_j \leq 0$,

$$f(x^*) \leq \min_{\substack{a_j' x \leq b_j \\ j \notin J}} \left\{ f(x) + \sum_{j=1}^{r} \mu_j^* (a_j' x - b_j) \right\}$$

$$\leq \min_{\substack{a_j' x \leq b_j \\ j \notin J}} \left\{ f(x) + \sum_{j \in J} \mu_j^* (a_j' x - b_j) \right\}.$$

PROOF (CONTINUED)

Conversely, if x^* is feasible and there exist scalars μ_j^*, $j \in J$ with the stated properties, then

$$\left(\nabla f(x^*) + \sum_{j \in J} \mu_j^* a_j\right)'(x - x^*) \geq 0, \quad \text{if } a_j' x \leq b_j, \ \forall \ j \notin J.$$

For all x that are feasible for the original problem, $a_j' x \leq b_j = a_j' x^*$ for all $j \in A(x^*)$. Since $\mu_j^* = 0$ if $j \in J$ and $j \notin A(x^*)$,

$$\sum_{j \in J} \mu_j^* a_j'(x - x^*) \leq 0,$$

which implies

$$\nabla f(x^*)'(x - x^*) \geq 0$$

for all feasible x. Hence x^* is a global min. **Q.E.D.**

- Note that the same set of μ_j^* works for all index sets J.

THE DUAL PROBLEM

- Consider the problem

$$\min_{x \in X,\, a_j' x \le b_j,\; j=1,\ldots,r} f(x)$$

where f is convex and cont. differentiable over $\Re^n$ and X is polyhedral.

- Define the *dual function* $q : \Re^r \mapsto [-\infty, \infty)$

$$q(\mu) = \inf_{x \in X} L(x, \mu) = \inf_{x \in X} \left\{ f(x) + \sum_{j=1}^{r} \mu_j (a_j' x - b_j) \right\}$$

and the *dual problem*

$$\max_{\mu \ge 0} q(\mu).$$

- If X is bounded, the dual function takes real values. In general, $q(\mu)$ can take the value $-\infty$. The "effective" constraint set of the dual is

$$Q = \left\{ \mu \mid \mu \ge 0,\; q(\mu) > -\infty \right\}.$$

DUALITY THEOREM

(a) If the primal problem has an optimal solution, the dual problem also has an optimal solution and the optimal values are equal.

(b) x^* is primal-optimal and μ^* is dual-optimal if and only if x^* is primal-feasible, $\mu^* \geq 0$, and

$$f(x^*) = L(x^*, \mu^*) = \min_{x \in X} L(x, \mu^*).$$

Proof: (a) Let x^* be a primal optimal solution. For all primal feasible x, and all $\mu \geq 0$, we have $\mu_j'(a_j'x - b_j) \leq 0$ for all j, so

$$\begin{aligned} q(\mu) &\leq \inf_{x \in X,\, a_j'x \leq b_j,\ j=1,\ldots,r} \left\{ f(x) + \sum_{j=1}^{r} \mu_j (a_j'x - b_j) \right\} \\ &\leq \inf_{x \in X,\, a_j'x \leq b_j,\ j=1,\ldots,r} f(x) = f(x^*). \end{aligned} \tag{*}$$

By L-Mult. Th., there exists $\mu^* \geq 0$ such that $\mu_j^*(a_j'x^* - b_j) = 0$ for all j, and $x^* = \arg\min_{x \in X} L(x, \mu^*)$, so

$$q(\mu^*) = L(x^*, \mu^*) = f(x^*) + \sum_{j=1}^{r} \mu_j^* (a_j'x^* - b_j) = f(x^*).$$

PROOF (CONTINUED)

(b) If x^* is primal-optimal and μ^* is dual-optimal, by part (a)

$$f(x^*) = q(\mu^*),$$

which when combined with Eq. (*), yields

$$f(x^*) = L(x^*, \mu^*) = q(\mu^*) = \min_{x \in X} L(x, \mu^*).$$

Conversely, the relation $f(x^*) = \min_{x \in X} L(x, \mu^*)$ is written as $f(x^*) = q(\mu^*)$, and since x^* is primal-feasible and $\mu^* \geq 0$, Eq. (*) implies that x^* is primal-optimal and μ^* is dual-optimal. **Q.E.D.**

- Linear equality constraints are treated similar to inequality constraints, except that the sign of the Lagrange multipliers is unrestricted:

Primal: $$\min_{x \in X,\, e_i' x = d_i,\, i=1,\ldots,m\ a_j' x \leq b_j,\ j=1,\ldots,r} f(x)$$

Dual: $$\max_{\lambda \in \Re^m,\, \mu \geq 0} q(\lambda, \mu) = \max_{\lambda \in \Re^m,\, \mu \geq 0} \inf_{x \in X} L(x, \lambda, \mu).$$

THE DUAL OF A LINEAR PROGRAM

- Consider the linear program

$$\begin{aligned} &\text{minimize} \quad c'x \\ &\text{subject to} \quad e_i'x = d_i, \quad i = 1, \ldots, m, \qquad x \geq 0 \end{aligned}$$

- Dual function

$$q(\lambda) = \inf_{x \geq 0} \left\{ \sum_{j=1}^{n} \left(c_j - \sum_{i=1}^{m} \lambda_i e_{ij} \right) x_j + \sum_{i=1}^{m} \lambda_i d_i \right\}.$$

- If $c_j - \sum_{i=1}^m \lambda_i e_{ij} \geq 0$ for all j, the infimum is attained for $x = 0$, and $q(\lambda) = \sum_{i=1}^m \lambda_i d_i$. If $c_j - \sum_{i=1}^m \lambda_i e_{ij} < 0$ for some j, the expression in braces can be arbitrarily small by taking x_j suff. large, so $q(\lambda) = -\infty$. Thus, the dual is

$$\begin{aligned} &\text{maximize} \quad \sum_{i=1}^{m} \lambda_i d_i \\ &\text{subject to} \quad \sum_{i=1}^{m} \lambda_i e_{ij} \leq c_j, \qquad j = 1, \ldots, n. \end{aligned}$$

THE DUAL OF A QUADRATIC PROGRAM

- Consider the quadratic program

$$\text{minimize} \quad \tfrac{1}{2}x'Qx + c'x$$
$$\text{subject to} \quad Ax \le b,$$

where Q is a given $n \times n$ positive definite symmetric matrix, A is a given $r \times n$ matrix, and $b \in \Re^r$ and $c \in \Re^n$ are given vectors.

- Dual function:

$$q(\mu) = \inf_{x \in \Re^n} \left\{ \tfrac{1}{2}x'Qx + c'x + \mu'(Ax - b) \right\}.$$

The infimum is attained for $x = -Q^{-1}(c + A'\mu)$, and, after substitution and calculation,

$$q(\mu) = -\tfrac{1}{2}\mu' AQ^{-1}A'\mu - \mu'(b + AQ^{-1}c) - \tfrac{1}{2}c'Q^{-1}c.$$

- The dual problem, after a sign change, is

$$\text{minimize} \quad \tfrac{1}{2}\mu' P\mu + t'\mu$$
$$\text{subject to} \quad \mu \ge 0,$$

where $P = AQ^{-1}A'$ and $t = b + AQ^{-1}c$.