6.252 NONLINEAR PROGRAMMING

LECTURE 13: INEQUALITY CONSTRAINTS

LECTURE OUTLINE

- Inequality Constrained Problems
- Necessary Conditions
- Sufficiency Conditions
- Linear Constraints

Inequality constrained problem

$$\text{minimize } f(x)$$
$$\text{subject to } h(x) = 0, \qquad g(x) \le 0$$

where $f : \Re^n \mapsto \Re$, $h : \Re^n \mapsto \Re^m$, $g : \Re^n \mapsto \Re^r$ are continuously differentiable. Here

$$h = (h_1, ..., h_m), \qquad g = (g_1, ..., g_r).$$

TREATING INEQUALITIES AS EQUATIONS

- Consider the *set of active inequality constraints*

$$A(x) = \Big\{ j \mid g_j(x) = 0 \Big\}.$$

- If x^* is a local minimum:
  - The active inequality constraints at x^* can be treated as equations
  - The inactive constraints at x^* don't matter

- Assuming regularity of x^* and assigning zero Lagrange multipliers to inactive constraints,

$$\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla h_i(x^*) + \sum_{j=1}^{r} \mu_j^* \nabla g_j(x^*) = 0,$$

$$\mu_j^* = 0, \qquad \forall\ j \notin A(x^*).$$

- Extra property: $\mu_j^* \geq 0$ for all j.

Intuitive reason: Relax jth constraint, $g_j(x) \leq u_j$,

$$\mu_j^* = -(\Delta\text{cost due to } u_j)/u_j$$

BASIC RESULTS

Kuhn-Tucker Necessary Conditions: Let x^* be a local minimum and a regular point. Then there exist unique Lagrange mult. vectors $\lambda^* = (\lambda_1^*, \ldots, \lambda_m^*)$, $\mu^* = (\mu_1^*, \ldots, \mu_r^*)$, such that

$$\nabla_x L(x^*, \lambda^*, \mu^*) = 0,$$

$$\mu_j^* \geq 0, \qquad j = 1, \ldots, r,$$

$$\mu_j^* = 0, \qquad \forall\ j \notin A(x^*).$$

If f, h, and g are twice cont. differentiable,

$$y' \nabla_{xx}^2 L(x^*, \lambda^*, \mu^*) y \geq 0, \qquad \text{for all } y \in V(x^*),$$

where

$$V(x^*) = \big\{ y \mid \nabla h(x^*)' y = 0,\ \nabla g_j(x^*)' y = 0,\ j \in A(x^*) \big\}.$$

- Similar sufficiency conditions and sensitivity results. They require strict complementarity, i.e.,

$$\mu_j^* > 0, \qquad \forall\ j \in A(x^*).$$

PROOF OF KUHN-TUCKER CONDITIONS

Use equality-constraints result to obtain all the conditions except for $\mu_j^* \geq 0$ for $j \in A(x^*)$. Introduce the penalty functions

$$g_j^+(x) = \max\Big\{0, g_j(x)\Big\}, \qquad j = 1, \ldots, r,$$

and for $k = 1, 2, \ldots$, let x^k minimize

$$f(x) + \frac{k}{2}||h(x)||^2 + \frac{k}{2}\sum_{j=1}^{r}\big(g_j^+(x)\big)^2 + \frac{1}{2}||x - x^*||^2$$

over a closed sphere of x such that $f(x^*) \leq f(x)$. Using the same argument as for equality constraints,

$$\lambda_i^* = \lim_{k \to \infty} k h_i(x^k), \qquad i = 1, \ldots, m,$$

$$\mu_j^* = \lim_{k \to \infty} k g_j^+(x^k), \qquad j = 1, \ldots, r.$$

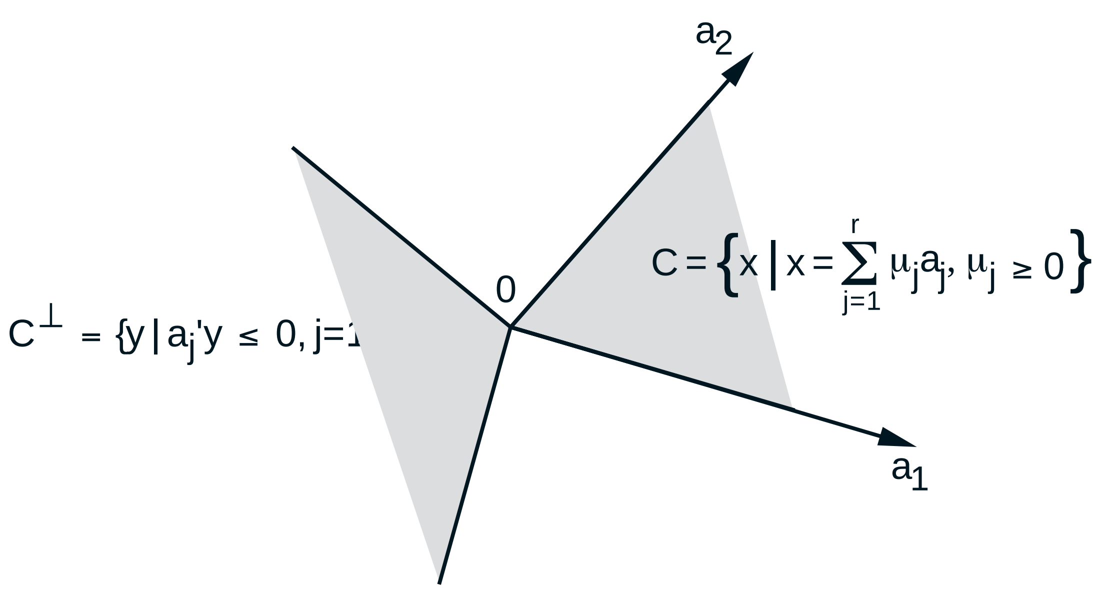
Since $g_j^+(x^k) \geq 0$, we obtain $\mu_j^* \geq 0$ for all j.

LINEAR CONSTRAINTS

- Consider the problem $\min_{a_j' x \leq b_j,\, j=1,\ldots,r} f(x)$.
- Remarkable property: No need for regularity.
- Proposition: If x^* is a local minimum, there exist $\mu_1^*, \ldots, \mu_r^*$ with $\mu_j^* \geq 0$, $j = 1, \ldots, r$, such that

$$\nabla f(x^*) + \sum_{j=1}^{r} \mu_j^* a_j = 0, \qquad \mu_j^* = 0, \quad \forall\, j \notin A(x^*).$$

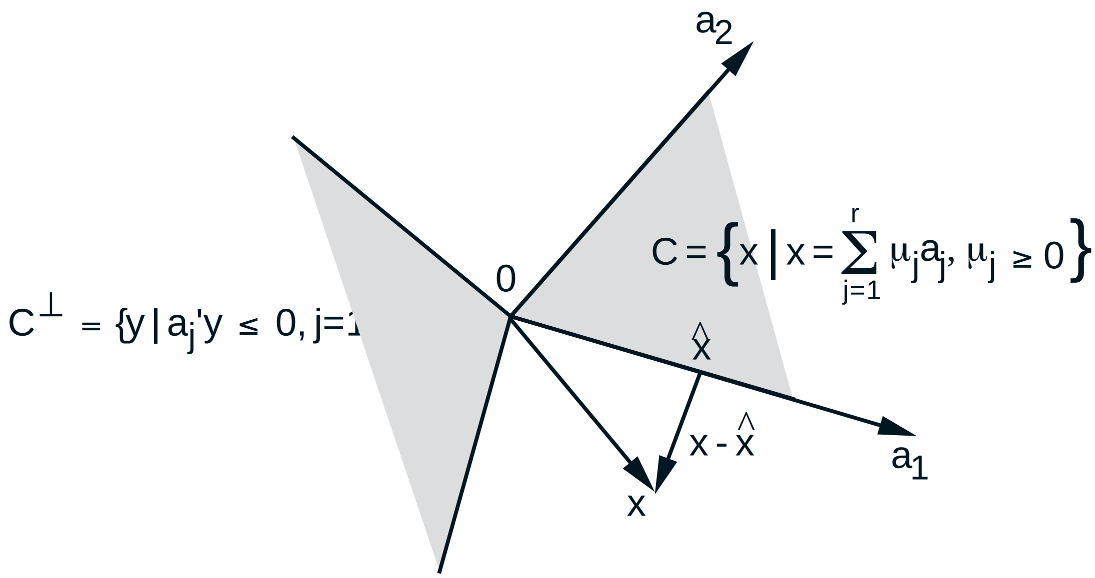
- Proof uses Farkas Lemma: Consider the cones C and $C^\perp$



$$x \in C \qquad \text{iff} \qquad x'y \leq 0, \quad \forall\, y \in C^\perp.$$

PROOF OF FARKAS LEMMA

$$x \in C \quad \text{iff} \quad x'y \leq 0, \quad \forall\, y \in C^\perp.$$



Proof: First show that C is closed (nontrivial). Then, let x be such that $x'y \leq 0, \forall\, y \in C^\perp$, and consider its projection $\hat{x}$ on C. We have

$$x'(x - \hat{x}) = \|x - \hat{x}\|^2, \qquad (*)$$

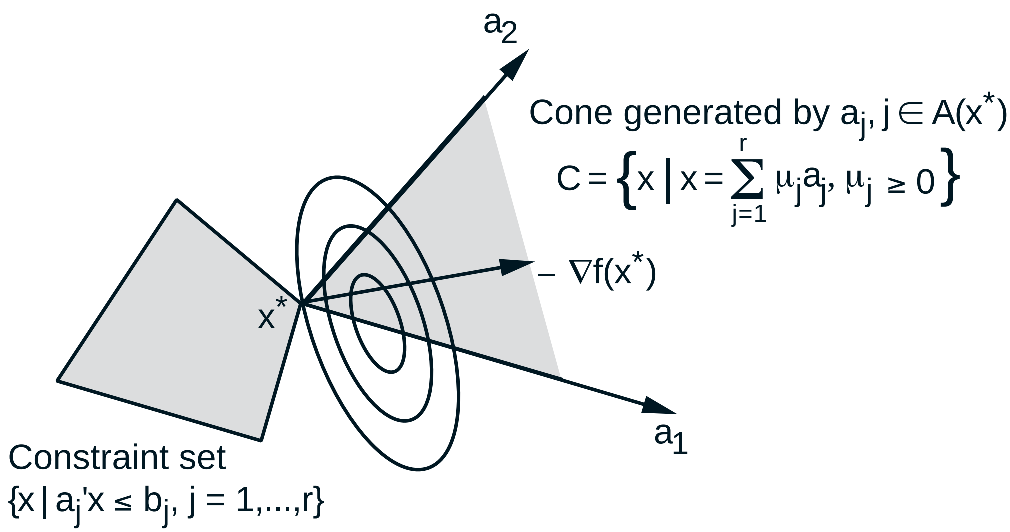
$$(x - \hat{x})' a_j \leq 0, \qquad \forall\, j.$$

Hence, $(x - \hat{x}) \in C^\perp$, and using the hypothesis,

$$x'(x - \hat{x}) \leq 0. \qquad (**)$$

From $(*)$ and $(**)$, we obtain $x = \hat{x}$, so $x \in C$.

PROOF OF LAGRANGE MULTIPLIER RESULT



The local min x^* of the original problem is also a local min for the problem $\min_{a_j' x \leq b_j,\, j \in A(x^*)} f(x)$. Hence

$$\nabla f(x^*)'(x - x^*) \geq 0, \quad \forall\, x \text{ with } a_j' x \leq b_j,\ j \in A(x^*).$$

Since a constraint $a_j' x \leq b_j,\ j \in A(x^*)$ can also be expressed as $a_j'(x - x^*) \leq 0$, we have

$$\nabla f(x^*)' y \geq 0, \qquad \forall\, y \text{ with } a_j' y \leq 0,\ j \in A(x^*).$$

From Farkas' lemma, $-\nabla f(x^*)$ has the form

$$\sum_{j \in A(x^*)} \mu_j^* a_j, \quad \text{for some } \mu_j^* \geq 0,\ j \in A(x^*).$$

Let $\mu_j^* = 0$ for $j \notin A(x^*)$.

CONVEX COST AND LINEAR CONSTRAINTS

Let $f : \Re^n \mapsto \Re$ be convex and cont. differentiable, and let J be a subset of the index set $\{1, \ldots, r\}$. Then x^* is a global minimum for the problem

$$\text{minimize } f(x)$$
$$\text{subject to } a_j' x \le b_j, \qquad j = 1, \ldots, r,$$

if and only if x^* is feasible and there exist scalars μ_j^*, $j \in J$, such that

$$\mu_j^* \ge 0, \qquad j \in J,$$

$$\mu_j^* = 0, \qquad \forall\ j \in J \text{ with } j \notin A(x^*),$$

$$x^* = \arg \min_{\substack{a_j' x \le b_j \\ j \notin J}} \left\{ f(x) + \sum_{j \in J} \mu_j^* (a_j' x - b_j) \right\}.$$

- Proof is immediate if $J = \{1, \ldots, r\}$.
- Example: Simplex Constraint.