6.252 NONLINEAR PROGRAMMING

LECTURE 12: SUFFICIENCY CONDITIONS

LECTURE OUTLINE

- Equality Constrained Problems/Sufficiency Conditions
- Convexification Using Augmented Lagrangians
- Proof of the Sufficiency Conditions
- Sensitivity

Equality constrained problem

$$\text{minimize } f(x)$$
$$\text{subject to } h_i(x) = 0, \qquad i = 1, \ldots, m.$$

where $f : \Re^n \mapsto \Re$, $h_i : \Re^n \mapsto \Re$, are continuously differentiable. To obtain sufficiency conditions, assume that f and h_i are *twice* continuously differentiable.

SUFFICIENCY CONDITIONS

Second Order Sufficiency Conditions: Let $x^* \in \Re^n$ and $\lambda^* \in \Re^m$ satisfy

$$\nabla_x L(x^*, \lambda^*) = 0, \qquad \nabla_\lambda L(x^*, \lambda^*) = 0,$$

$$y' \nabla^2_{xx} L(x^*, \lambda^*) y > 0, \quad \forall\ y \neq 0 \text{ with } \nabla h(x^*)' y = 0.$$

Then x^* is a strict local minimum.

Example: Minimize $-(x_1 x_2 + x_2 x_3 + x_1 x_3)$ subject to $x_1 + x_2 + x_3 = 3$. We have that $x_1^* = x_2^* = x_3^* = 1$ and $\lambda^* = 2$ satisfy the 1st order conditions. Also

$$\nabla^2_{xx} L(x^*, \lambda^*) = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}.$$

We have for all $y \neq 0$ with $\nabla h(x^*)' y = 0$ or $y_1 + y_2 + y_3 = 0$,

$$\begin{aligned} y' \nabla^2_{xx} L(x^*, \lambda^*) y &= -y_1(y_2 + y_3) - y_2(y_1 + y_3) - y_3(y_1 + y_2) \\ &= y_1^2 + y_2^2 + y_3^2 > 0. \end{aligned}$$

Hence, x^* is a strict local minimum.

A BASIC LEMMA

Lemma: Let P and Q be two symmetric matrices. Assume that $Q \geq 0$ and $P > 0$ on the nullspace of Q, i.e., $x'Px > 0$ for all $x \neq 0$ with $x'Qx = 0$. Then there exists a scalar $\overline{c}$ such that

$$P + cQ : \text{positive definite}, \qquad \forall\ c > \overline{c}.$$

Proof: Assume the contrary. Then for every k, there exists a vector x^k with $\|x^k\| = 1$ such that

$$x^{k'} P x^k + k x^{k'} Q x^k \leq 0.$$

Consider a subsequence $\{x^k\}_{k \in K}$ converging to some $\overline{x}$ with $\|\overline{x}\| = 1$. Taking the limit superior,

$$\overline{x}' P \bar{x} + \limsup_{k \to \infty,\, k \in K} (k x^{k'} Q x^k) \leq 0. \qquad (*)$$

We have $x^{k'} Q x^k \geq 0$ (since $Q \geq 0$), so $\{x^{k'} Q x^k\}_{k \in K} \to 0$. Therefore, $\overline{x}' Q \overline{x} = 0$ and using the hypothesis, $\overline{x}' P \overline{x} > 0$. This contradicts (*).

PROOF OF SUFFICIENCY CONDITIONS

Consider the *augmented Lagrangian* function

$$L_c(x, \lambda) = f(x) + \lambda' h(x) + \frac{c}{2}\|h(x)\|^2,$$

where c is a scalar. We have

$$\nabla_x L_c(x, \lambda) = \nabla_x L(x, \tilde{\lambda}),$$

$$\nabla^2_{xx} L_c(x, \lambda) = \nabla^2_{xx} L(x, \tilde{\lambda}) + c \nabla h(x) \nabla h(x)'$$

where $\tilde{\lambda} = \lambda + ch(x)$. If (x^*, λ^*) satisfy the suff. conditions, we have using the lemma,

$$\nabla_x L_c(x^*, \lambda^*) = 0, \qquad \nabla^2_{xx} L_c(x^*, \lambda^*) > 0,$$

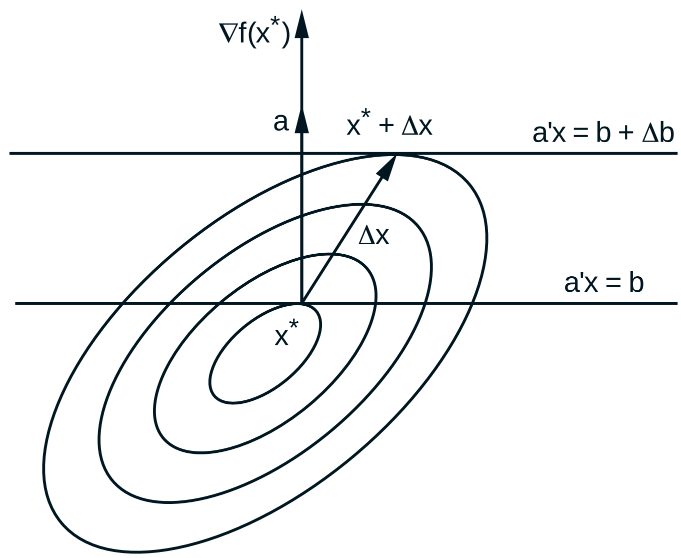
for suff. large c. Hence for some $\gamma > 0$, $\epsilon > 0$,

$$L_c(x, \lambda^*) \geq L_c(x^*, \lambda^*) + \frac{\gamma}{2}\|x - x^*\|^2, \quad \text{if } \|x - x^*\| < \epsilon.$$

Since $L_c(x, \lambda^*) = f(x)$ when $h(x) = 0$,

$$f(x) \geq f(x^*) + \frac{\gamma}{2}\|x - x^*\|^2, \quad \text{if } h(x) = 0,\ \|x - x^*\| < \epsilon.$$

SENSITIVITY - GRAPHICAL DERIVATION



Sensitivity theorem for the problem $\min_{a'x=b} f(x)$. If b is changed to $b+\Delta b$, the minimum x^* will change to $x^*+\Delta x$. Since $b+\Delta b = a'(x^*+\Delta x) = a'x^* + a'\Delta x = b + a'\Delta x$, we have $a'\Delta x = \Delta b$. Using the condition $\nabla f(x^*) = -\lambda^* a$,

$$\begin{aligned}\Delta\text{cost} &= f(x^*+\Delta x) - f(x^*) = \nabla f(x^*)'\Delta x + o(\|\Delta x\|)\\ &= -\lambda^* a'\Delta x + o(\|\Delta x\|)\end{aligned}$$

Thus $\Delta\text{cost} = -\lambda^*\Delta b + o(\|\Delta x\|)$, so up to first order

$$\lambda^* = -\frac{\Delta\text{cost}}{\Delta b}.$$

For multiple constraints $a_i'x = b_i$, $i = 1,\ldots,n$, we have

$$\Delta\text{cost} = -\sum_{i=1}^{m}\lambda_i^*\Delta b_i + o(\|\Delta x\|).$$

SENSITIVITY THEOREM

Sensitivity Theorem: Consider the family of problems

$$\min_{h(x)=u} f(x) \qquad (*)$$

parameterized by $u \in \Re^m$. Assume that for $u = 0$, this problem has a local minimum x^*, which is regular and together with its unique Lagrange multiplier λ^* satisfies the sufficiency conditions.

Then there exists an open sphere S centered at $u = 0$ such that for every $u \in S$, there is an $x(u)$ and a $\lambda(u)$, which are a local minimum-Lagrange multiplier pair of problem (*). Furthermore, $x(\cdot)$ and $\lambda(\cdot)$ are continuously differentiable within S and we have $x(0) = x^*$, $\lambda(0) = \lambda^*$. In addition,

$$\nabla p(u) = -\lambda(u), \qquad \forall\ u \in S$$

where $p(u)$ is the *primal function*

$$p(u) = f\big(x(u)\big).$$

EXAMPLE

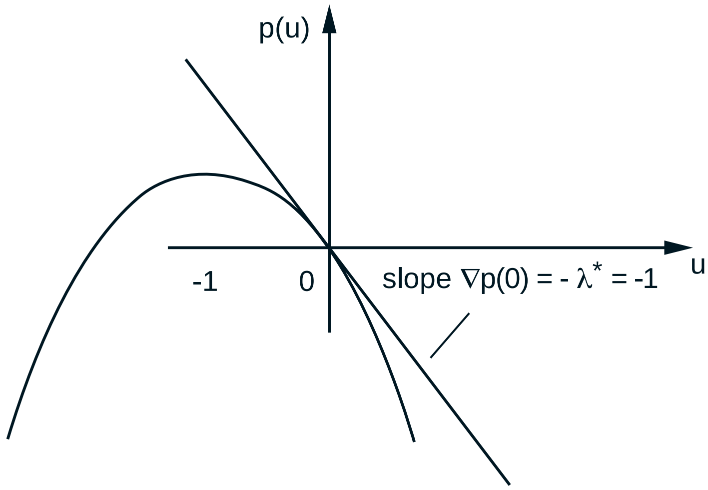


Illustration of the primal function $p(u) = f\big(x(u)\big)$ for the two-dimensional problem

$$\text{minimize} \;\; f(x) = \tfrac{1}{2}\left(x_1^2 - x_2^2\right) - x_2$$

$$\text{subject to} \;\; h(x) = x_2 = 0.$$

Here,

$$p(u) = \min_{h(x)=u} f(x) = -\tfrac{1}{2}u^2 - u$$

and $\lambda^* = -\nabla p(0) = 1$, consistently with the sensitivity theorem.

- Need for regularity of x^*: Change constraint to $h(x) = x_2^2 = 0$. Then $p(u) = -u/2 - \sqrt{u}$ for $u \geq 0$ and is undefined for $u < 0$.

PROOF OUTLINE OF SENSITIVITY THEOREM

Apply implicit function theorem to the system

$$\nabla f(x) + \nabla h(x)\lambda = 0, \qquad h(x) = u.$$

For $u = 0$ the system has the solution (x^*, λ^*), and the corresponding $(n+m) \times (n+m)$ Jacobian

$$J = \begin{pmatrix} \nabla^2 f(x^*) + \sum_{i=1}^m \lambda_i^* \nabla^2 h_i(x^*) & \nabla h(x^*) \\ \nabla h(x^*)' & 0 \end{pmatrix}$$

is shown nonsingular using the sufficiency conditions. Hence, for all u in some open sphere S centered at $u = 0$, there exist $x(u)$ and $\lambda(u)$ such that $x(0) = x^*$, $\lambda(0) = \lambda^*$, the functions $x(\cdot)$ and $\lambda(\cdot)$ are continuously differentiable, and

$$\nabla f\big(x(u)\big) + \nabla h\big(x(u)\big)\lambda(u) = 0, \quad h\big(x(u)\big) = u.$$

For u close to $u = 0$, using the sufficiency conditions, $x(u)$ and $\lambda(u)$ are a local minimum-Lagrange multiplier pair for the problem $\min_{h(x)=u} f(x)$.

To derive $\nabla p(u)$, differentiate $h\big(x(u)\big) = u$, to obtain $I = \nabla x(u) \nabla h\big(x(u)\big)$, and combine with the relations $\nabla x(u)\nabla f\big(x(u)\big) + \nabla x(u) \nabla h\big(x(u)\big)\lambda(u) = 0$ and $\nabla p(u) = \nabla_u \Big\{ f\big(x(u)\big)\Big\} = \nabla x(u) \nabla f\big(x(u)\big)$.