6.252 NONLINEAR PROGRAMMING

LECTURE 2

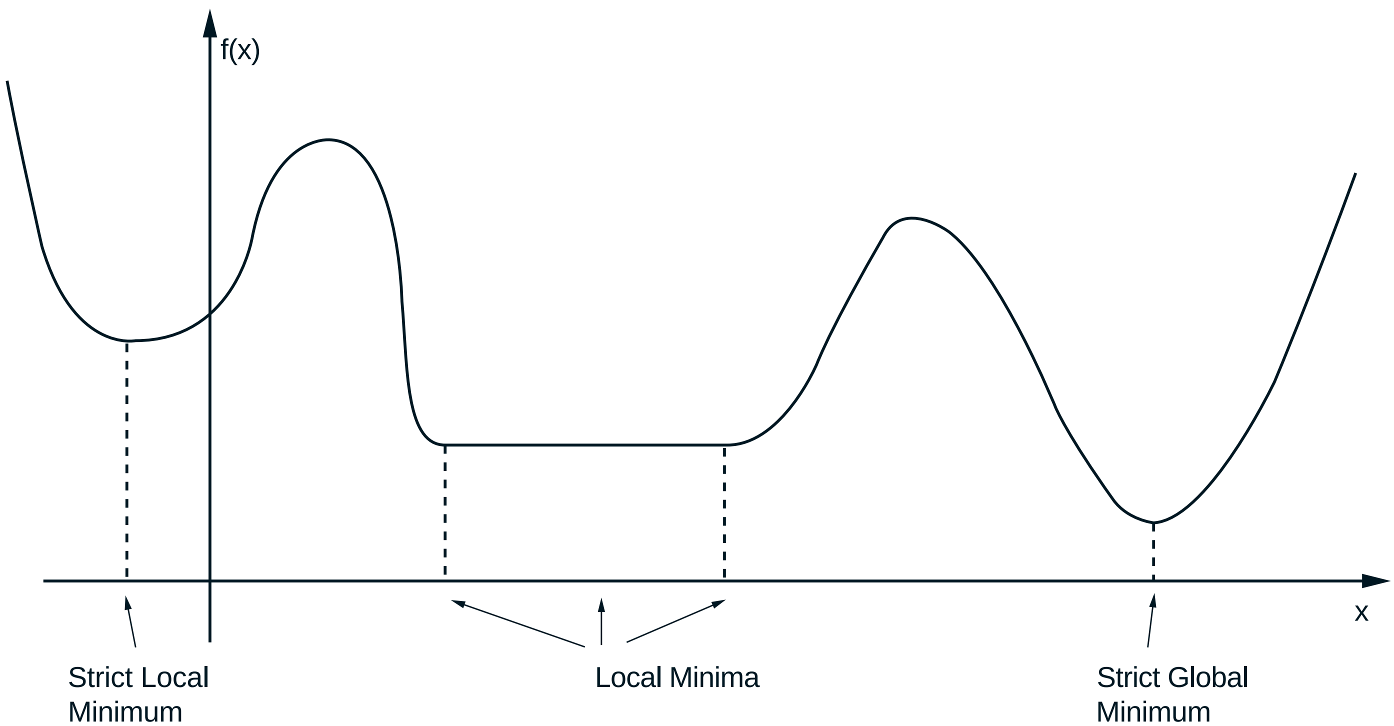
UNCONSTRAINED OPTIMIZATION -

OPTIMALITY CONDITIONS

LECTURE OUTLINE

- Unconstrained Optimization
- Local Minima
- Necessary Conditions for Local Minima
- Sufficient Conditions for Local Minima
- The Role of Convexity

LOCAL AND GLOBAL MINIMA



Unconstrained local and global minima in one dimension.

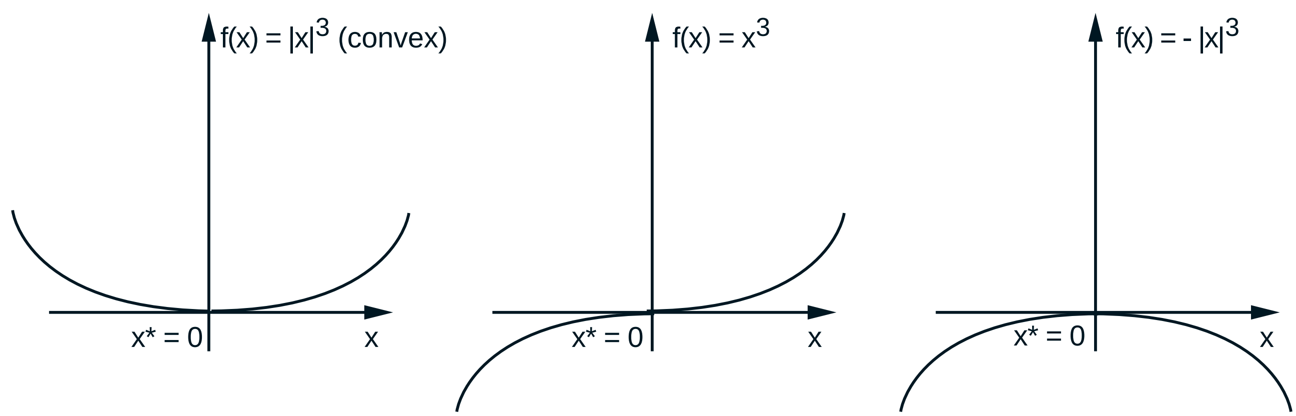
NECESSARY CONDITIONS FOR A LOCAL MIN

- Zero slope at a local minimum x^*

$$\nabla f(x^*) = 0$$

- Nonnegative curvature at a local minimum x^*

$$\nabla^2 f(x^*) : \text{ Positive Semidefinite}$$



First and second order necessary optimality conditions for functions of one variable.

PROOFS OF NECESSARY CONDITIONS

- 1st order condition $\nabla f(x^*) = 0$. Fix $d \in \Re^n$. Then (since x^* is a local min)

$$d'\nabla f(x^*) = \lim_{\alpha \downarrow 0} \frac{f(x^* + \alpha d) - f(x^*)}{\alpha} \geq 0,$$

Replace d with $-d$, to obtain

$$d'\nabla f(x^*) = 0, \quad \forall\, d \in \Re^n$$

- 2nd order condition $\nabla^2 f(x^*) \geq 0$.

$$f(x^*+\alpha d)-f(x^*) = \alpha \nabla f(x^*)'d + \frac{\alpha^2}{2} d'\nabla^2 f(x^*)d + o(\alpha^2)$$

Since $\nabla f(x^*) = 0$ and x^* is local min, there is sufficiently small $\epsilon > 0$ such that for all $\alpha \in (0, \epsilon)$,

$$0 \leq \frac{f(x^* + \alpha d) - f(x^*)}{\alpha^2} = \tfrac{1}{2} d'\nabla^2 f(x^*)d + \frac{o(\alpha^2)}{\alpha^2}$$

Take the limit as $\alpha \to 0$.

SUFFICIENT CONDITIONS FOR A LOCAL MIN

- Zero slope

$$\nabla f(x^*) = 0$$

- Positive curvature

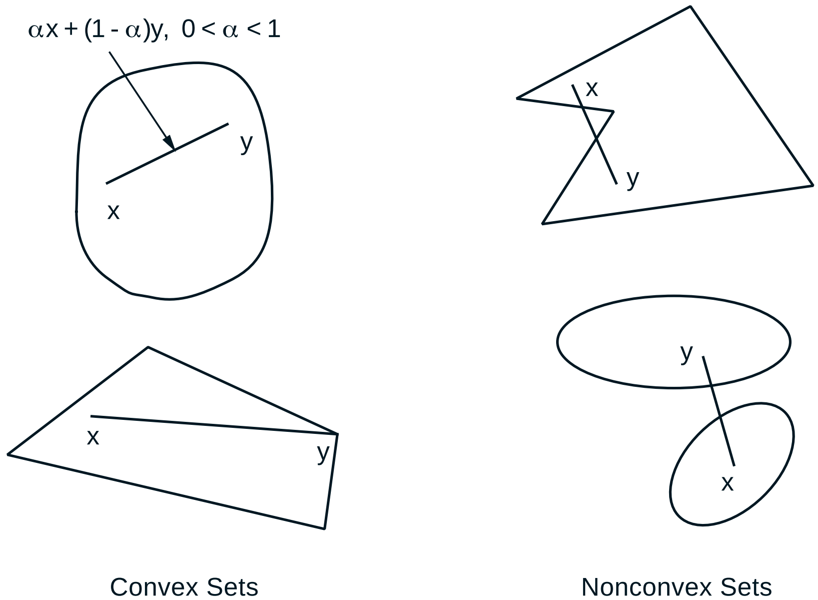
$$\nabla^2 f(x^*) : \text{Positive Definite}$$

- Proof: Let $\lambda > 0$ be the smallest eigenvalue of $\nabla^2 f(x^*)$. Using a second order Taylor expansion, we have for all d

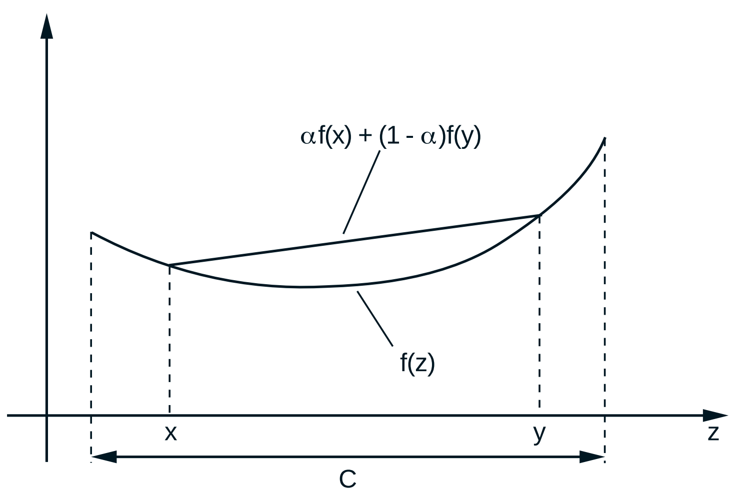
$$\begin{aligned} f(x^* + d) - f(x^*) &= \nabla f(x^*)' d + \frac{1}{2} d' \nabla^2 f(x^*) d \\ &\quad + o(\|d\|^2) \\ &\geq \frac{\lambda}{2}\|d\|^2 + o(\|d\|^2) \\ &= \left(\frac{\lambda}{2} + \frac{o(\|d\|^2)}{\|d\|^2}\right)\|d\|^2. \end{aligned}$$

For $\|d\|$ small enough, $o(\|d\|^2)/\|d\|^2$ is negligible relative to $\lambda/2$.

CONVEXITY



Convex and nonconvex sets.



A convex function.

MINIMA AND CONVEXITY

- Local minima are also global under convexity

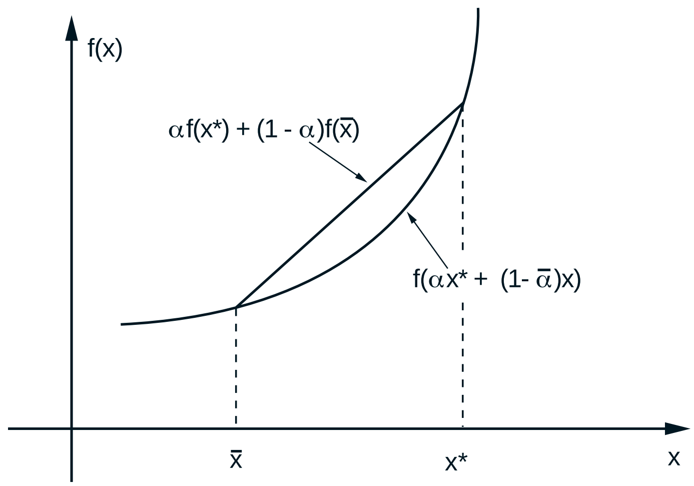


Illustration of why local minima of convex functions are also global. Suppose that f is convex and that x^* is a local minimum of f. Let $\overline{x}$ be such that $f(\overline{x}) < f(x^*)$. By convexity, for all $\alpha \in (0,1)$,

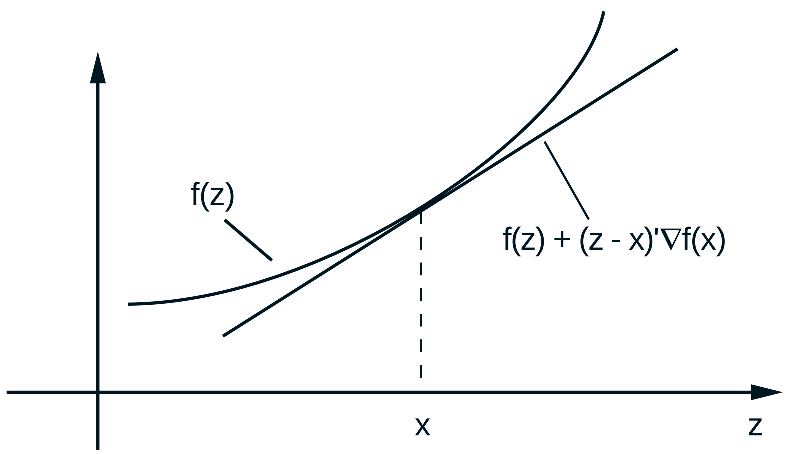
$$f\Big(\alpha x^* + (1-\alpha)\overline{x}\Big) \leq \alpha f(x^*) + (1-\alpha) f(\overline{x}) < f(x^*).$$

Thus, f takes values strictly lower than $f(x^*)$ on the line segment connecting x^* with $\overline{x}$, and x^* cannot be a local minimum which is not global.

OTHER PROPERTIES OF CONVEX FUNCTIONS

- f is convex if and only if the linear approximation at a point x^* based on the gradient, that is,

$$f(x) \geq f(x^*) + \nabla f(x^*)'(x - x^*), \quad \forall\, x$$



  – Implication:

$$\nabla f(x^*) = 0 \quad \Rightarrow x^* \text{ is a global minimum}$$

- f is convex if and only if $\nabla^2 f(x)$ is positive semidefinite for all x