15.081J/6.251J Introduction to Mathematical Programming

Lecture 19: Problems with exponentially many constraints

1 Outline

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- Problems with exponentially many constraints
- The separation problem
- Polynomial solvability
- Examples: MST, TSP, Probability
- Conclusions

2 Problems

2.1 Example

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$$\min \sum_i c_i x_i$$

$$\sum_{i \in S} a_i x_i \geq |S|, \qquad \text{for all subsets } S \text{ of } \{1, \ldots, n\}$$

- There are 2^n constraints, but are described concisely in terms of the n scalar parameters $a_1, \ldots, a_n$
- Question: Suppose we apply the ellipsoid algorithm. Is it polynomial?
- In what?

2.2 The input

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- Consider $\min \boldsymbol{c}'\boldsymbol{x}$ s.t. $\boldsymbol{x} \in P$
- P belongs to a family of polyhedra of special structure
- A typical polyhedron is described by specifying the dimension n and an integer vector $\boldsymbol{h}$ of *primary data*, of dimension $O(n^k)$, where $k \geq 1$ is some constant.
- In example, $\boldsymbol{h} = (a_1, \ldots, a_n)$ and $k = 1$
- U_0 be the largest entry of $\boldsymbol{h}$

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- Given n and $\boldsymbol{h}$, P is described as $\boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}$
- $\boldsymbol{A}$ has an arbitrary number of rows
- U largest entry in $\boldsymbol{A}$ and $\boldsymbol{b}$. We assume

$$\log U \leq C n^{\ell} \log^{\ell} U_0$$

3 The separation problem

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Given a polyhedron $P \subset \Re^n$ and a vector $\boldsymbol{x} \in \Re^n$, the **separation problem** is to:

- Either decide that $\boldsymbol{x} \in P$, or
- Find a vector $\boldsymbol{d}$ such that $\boldsymbol{d}'\boldsymbol{x} < \boldsymbol{d}'\boldsymbol{y}$ for all $\boldsymbol{y} \in P$

What is the separation problem for

$$\sum_{i \in S} a_i x_i \geq |S|, \qquad \text{for all subsets } S \text{ of } \{1, \ldots, n\}?$$

4 Polynomial solvability

4.1 Theorem

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If we can solve the separation problem (for a family of polyhedra) in time polynomial in n and $\log U$, then we can also solve linear optimization problems in time polynomial in n and $\log U$. If $\log U \leq Cn^\ell \log^\ell U_0$, then it is also polynomial in $\log U_0$

- Proof ?
- Converse is also true
- Separation and optimization are polynomially equivalent

4.2 Minimum Spanning Tree (MST)

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- How do telephone companies bill you?
- It used to be that rate/minute: Boston $\rightarrow$ LA proportional to distance in MST
- Other applications: Telecommunications, Transportation (good lower bound for TSP)

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- Given a graph $G = (V, E)$ undirected and Costs c_e, $e \in E$.
- Find a tree of minimum cost spanning all the nodes.
- Decision variables $x_e = \begin{cases} 1, & \text{if edge } e \text{ is included in the tree} \\ 0, & \text{otherwise} \end{cases}$

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- The tree should be connected. How can you model this requirement?

- Let S be a set of vertices. Then S and $V \setminus S$ should be connected
- Let $\delta(S) = \{e = (i,j) \in E : \begin{array}{l} i \in S \\ j \in V \setminus S \end{array}\}$
- Then,
$$\sum_{e \in \delta(S)} x_e \geq 1$$
- What is the number of edges in a tree?
- Then, $\sum_{e \in E} x_e = n - 1$

4.2.1 Formulation

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$$\begin{aligned} IZ_{MST} = \quad & \min \sum_{e \in E} c_e x_e \\ H \quad & \left\{ \begin{array}{ll} \sum_{e \in \delta(S)} x_e \geq 1 & \forall\, S \subseteq V, S \neq \emptyset, V \\ \sum_{e \in E} x_e = n - 1 & \\ x_e \in \{0,1\}. & \end{array} \right. \end{aligned}$$

How can you solve the LP relaxation?

4.3 The Traveling Salesman Problem

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Given $G = (V, E)$ an undirected graph. $V = \{1, \ldots, n\}$, costs $c_e\ \forall\, e \in E$. Find a tour that minimizes total length.

4.3.1 Formulation

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$$x_e = \begin{cases} 1, & \text{if edge } e \text{ is included in the tour.} \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{array}{ll} \min & \sum_{e \in E} c_e x_e \\ \text{s.t.} & \sum_{e \in \delta(S)} x_e \geq 2, \quad S \subseteq E \\ & \sum_{e \in \delta(i)} x_e = 2, \quad i \in V \\ & x_e \in \{0,1\} \end{array}$$

How can you solve the LP relaxation?

4.4 Probability Theory

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- Events A_1, A_2
- $P(A_1) = 0.5,\ P(A_2) = 0.7,\ P(A_1 \cap A_2) \leq 0.1$
- Are these beliefs consistent?
- General problem: Given n events A_i $i \in N = \{1, \ldots, n\}$, beliefs

$$\mathrm{P}(A_i) \leq p_i, \qquad i \in N,$$
$$\mathrm{P}(A_i \cap A_j) \geq p_{ij}, \qquad i, j \in N,\ i < j.$$

- Given the numbers p_i and p_{ij}, which are between 0 and 1, are these beliefs consistent?

4.4.1 Formulation

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$$x(S) = \mathrm{P}\Big(\big(\cap_{i \in S} A_i\big) \cap \big(\cap_{i \notin S} \overline{A}_i\big)\Big),$$

$$\begin{aligned}
\sum_{\{S \mid i \in S\}} x(S) &\leq p_i, && i \in N, \\
\sum_{\{S \mid i,j \in S\}} x(S) &\geq p_{ij}, && i, j \in N,\ i < j, \\
\sum_{S} x(S) &= 1, && \\
x(S) &\geq 0, && \forall\ S.
\end{aligned}$$

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The previous LP is feasible if and only if there does not exist a vector $(\boldsymbol{u}, \boldsymbol{y}, z)$ such that

$$\begin{aligned}
\sum_{i,j \in S, i<j} y_{ij} + \sum_{i \in S} u_i + z &\geq 0, && \forall\ S, \\
\sum_{i,j \in N, i<j} p_{ij} y_{ij} + \sum_{i \in N} p_i u_i + z &\leq -1, && \\
y_{ij} \leq 0,\ u_i \geq 0, & && i, j \in N,\ i < j.
\end{aligned}$$

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Separation problem:

$$z^* + \min_S f(S) = \sum_{i,j \in S, i<j} y^*_{ij} + \sum_{i \in S} u^*_i \geq 0?$$

Example: $y^*_{12} = -2$, $y^*_{13} = -4$, $y^*_{14} = -4$, $y^*_{23} = -4$, $y^*_{24} = -1$, $y^*_{34} = -7$, $u^*_1 = 9$, $u^*_2 = 6$, $u^*_3 = 4$, $u^*_4 = 2$, and $z^* = 2$

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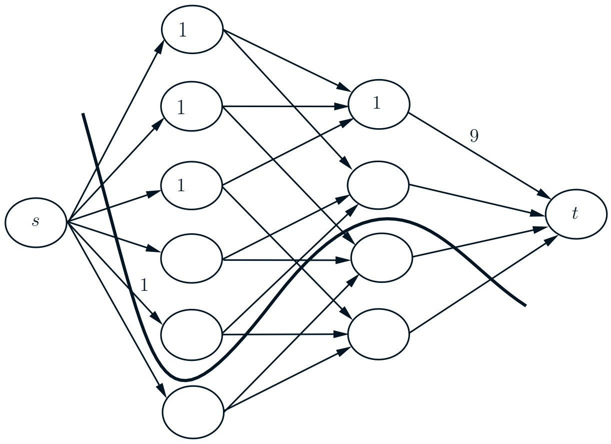
- The minimum cut corresponds to $S_0 = \{3, 4\}$ with value $c(S_0) = 21$.
- $f(S_0) = \sum_{i,j \in S_0, i<j} y^*_{ij} + \sum_{i \in S_0} u^*_i = -7 + 4 + 2 = -1$
- $f(S) + z^* \geq f(S_0) + z^* = -1 + 2 = 1 > 0, \qquad \forall\ S$
- Given solution $(\boldsymbol{y}^*, \boldsymbol{u}^*, z^*)$ is feasible



5 Conclusions

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- Ellipsoid algorithm can characterize the complexity of solving LOPs with an exponential number of constraints
- For practical purposes use dual simplex
- Ellipsoid method is an important theoretical development, not a practical one

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