15.081J/6.251J Introduction to Mathematical Programming

Lecture 16: Network Flows, I

1 Networks

Slide 1

- Electrical & Power Networks
- Road Networks
- Airline Routes
- Internet Backbone
- Printed Circuit Board
- Social Networks

2 Common Thrust

Slide 2

Move some entity (electricity, a consumer product, a person, a vehicle, a message, ...) from one point to another in the underlying network, as efficiently as possible.

1. Learn how to model application settings as network flow problems.
2. Study ways to solve the resulting models.

3 Shortest Path

3.1 Description

Slide 3

- Identify a shortest path from a given source node to a given sink node.
- Finding a path of minimum length.
- Finding a path taking minimum time.
- Finding a path of maximum reliability.

4 Maximum Flow

4.1 Description

Slide 4

- Determine the maximum flow that can be sent from a given source node to a sink node in a capacitated network.
- Determining maximum steady-state flow of
  - petroleum products in a pipeline network
  - cars in a road network
  - messages in a telecommunication network
  - electricity in an electrical network

5 Min-Cost Flow

5.1 Description

SLIDE 5

- Determine a least cost shipment of a commodity through a network in order to satisfy demands at certain nodes from available supplies at other nodes. Arcs have capacities and cost associated with them
- Distribution of products
- Flow of items in a production line
- Routing of cars through street networks
- Routing of telephone calls

5.2 In LOP Form

SLIDE 6

- Network $G = (N, A)$.
- Arc costs $c : A \to \mathcal{R}$.
- Arc capacities $u : A \to \mathcal{N}$.
- Node balances $b : N \to \mathcal{R}$.

$$\begin{array}{llrcrl}
\min & & \displaystyle\sum_{(i,j)\in A} c_{ij}x_{ij} & & & \\
\text{s.t.} & \displaystyle\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} & & = & b_i & \text{for all } i \in N \\
 & & x_{ij} & \leq & u_{ij} & \text{for all } (i,j) \in A \\
 & & x_{ij} & \geq & 0 & \text{for all } (i,j) \in A
\end{array}$$

6 Outline

SLIDE 7

- Shortest path applications
- Maximum Flow applications
- Minimum cost flow applications

7 Shortest Path

7.1 Interword Spacing in LaTeX

SLIDE 8

The spacing between words and characters is normally set automatically by LaTeX. Interword spacing within one line is uniform. LaTeX also attempts to keep the word spacing for different lines as nearly the same as possible.

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7.2 Interword Spacing in LaTeX (2)

SLIDE 9

- The paragraph consists of n words, indexed by $1, 2, \ldots, n$.
- c_{ij} is the attractiveness of a line if it begins with i and ends with $j - 1$.
- (LaTeX uses a formula to compute the value of each c_{ij}.)

For instance,

$$\begin{aligned} c_{12} &= -10,000 & c_{13} &= -1,000 \\ c_{14} &= 100 & c_{1,37} &= -100,000 \\ & \ldots \end{aligned}$$

7.3 Interword Spacing in LaTeX (3)

SLIDE 10

- The problem of decomposing a paragraph into several lines of text to maximize total attractiveness can be formulated as a shortest path problem.
- Nodes? Arcs? Costs?

7.4 Project Management

SLIDE 11

- A project consists of a set of jobs and a set of precedence relations
- Given a set A of job pairs (i, j) indicating that job i cannot start before job j is completed.
- c_i duration of job i
- Find the least possible duration of the project

7.4.1 Formulation

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- Introduce two artificial jobs s and t, of zero duration, that signify the beginning and the completion of the project
- Add (s, i) and (i, t) to A
- p_i time that job i begins
- $(i, j) \in A$: $p_j \geq p_i + c_i$
- Project duration: $p_t - p_s$

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- $$\begin{aligned} \min \quad & p_t - p_s \\ \text{s.t} \quad & p_j - p_i \geq c_i, \qquad \forall\ (i,j) \in A. \end{aligned}$$

- Dual

$$\begin{aligned} \max \quad & \sum_{(i,j)\in A} c_i f_{ij} \\ \text{s.t.} \quad & \sum_{\{j|(j,i)\in A\}} f_{ji} - \sum_{\{j|(i,j)\in A\}} f_{ij} = b_i \\ & f_{ij} \geq 0 \end{aligned}$$

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- $b_s = -1$, $b_t = 1$, and $b_i = 0$ for $i \neq s, t$.
- Shortest path problem, where each precedence relation $(i,j) \in A$ corresponds to an arc with cost of $-c_i$.

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Activity	Immediate Predecessor	Time(c_i)
s		0
A	S	14
B	S	3
C	A,B	5
D	A	7
E	C,D	10
t	E	0

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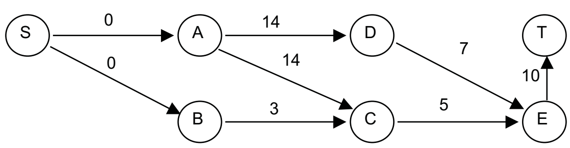

7.5 DNA Sequencing

Slide 17

- Given two sequences of letters, say

$$B = b_1 \cdots b_p \text{ and } D = d_1 \cdots d_q$$

- **How similar are the two sequences?**
- **What is the min cost of transforming B to D?**

7.5.1 Transformation costs

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- α = cost of inserting a letter in B
- β = cost of deleting a letter from B
- $g(b_i, d_j)$ = cost of mutating a letter b_i into d_j

7.5.2 Transformation steps

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1. Add or delete letters from B so as to make $|B'| = |D|$.
2. Align B' and D
3. Mutate letters of B' so that $B'' = D$.

7.5.3 Algorithm

SLIDE 20

- $f(b_1 \cdots b_p, d_1 \cdots d_q)$: the min cost of transforming B into D by the three steps above. We obtain this cost by a recursive way.
-

$$f(\emptyset \cdots \emptyset, d_1 \cdots d_j) = j\alpha, \quad j = 1, ..., q$$
$$f(b_1 \cdots b_i, \emptyset \cdots \emptyset) = i\beta, \quad i = 1, ..., p.$$

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Substitution

$B' =$	b_1	$\cdots$	b_i
$D =$	d_1	$\cdots$	d_j

$$\begin{aligned} & f(b_1 \cdots b_i, d_1 \cdots d_j) \\ = {} & f(b_1 \cdots b_{i-1}, d_1 \cdots d_{j-1}) + g(b_i, d_j) \end{aligned}$$

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- Addition of d_j

$B' =$	b_1	$\cdots$	b_i	$\cdots$	$\emptyset$
$D =$	d_1	$\cdots$		$\cdots$	d_j

$$f(b_1 \cdots b_i, d_1 \cdots d_j) = f(b_1 \cdots b_i, d_1 \cdots d_{j-1}) + \alpha.$$

- Deletion of b_i:

$$f(b_1 \cdots b_i, d_1 \cdots d_j) = f(b_1 \cdots b_{i-1}, d_1 \cdots d_j) + \beta$$

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Recursion

$$\begin{aligned} & f(b_1 \cdots b_i, d_1 \cdots d_j) \\ = \; & \min\{f(b_1 \cdots b_{i-1}, d_1 \cdots d_{j-1}) + g(b_i, d_j), \\ & \quad f(b_1 \cdots b_i, d_1 \cdots d_{j-1}) + \alpha, \\ & \quad f(b_1 \cdots b_{i-1}, d_1 \cdots d_j) + \beta\} \end{aligned}$$

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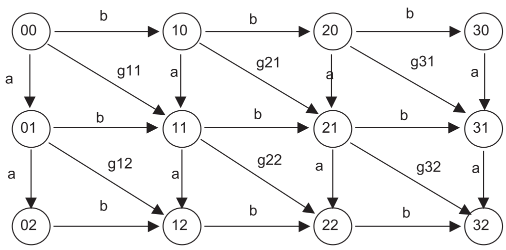
The shortest path from 00 to 32



8 Maximum Flow

8.1 The tournament problem

Slide 25

- Each of n teams plays against every other team a total of k games.
- Each game ends in a win or a loss (no draws)
- x_i: the number of wins of team i.
- X set of all possible outcome vectors $(x_1, ..., x_n)$
- Given $\boldsymbol{x} = (x_1, ..., x_n)$ decide whether $\boldsymbol{x} \in X$

8.1.1 Formulation

Slide 26

- Supply nodes $T_1, ..., T_n$ represent teams with supply $x_1, ..., x_n$
- Since total number of wins total number of games, we must have

$$\sum x_i = n(n-1)k/2$$

- Demand nodes

$$G_{12}, ..., G_{1n}, G_{23}, ..., G_{2n}, ..., G_{ij}, ..., G_{n-1,n}$$

  denote games between T_i and T_j with demand k.
- Arcs: (T_i, G_{ij}), (T_j, G_{ij}). The flow from T_i to G_{ij} represents the total number of games between i and j won by i
- Transportation model feasible if and only if $\boldsymbol{x} \in X$

8.2 Preemptive Scheduling

Slide 27

- m identical machines to process n jobs
- Job j must be processed for p_j periods, $j = 1, ..., n$
- It can not start before period r_j and must be competed before period d_j
- We allow preemption, i.e., we can disrupt the processing of one job with another

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- **Problem** Find a schedule (which job is processed by which machine at which period) such that all jobs are processed after their release times and completed before their deadlines
- C_j: completion time of job j: We need to have

$$r_j + p_j \leq C_j \leq d_j \text{ for all } j$$

8.3 Formulation

Slide 29

- Rank all release times and deadlines in ascending order. The ordered list of numbers divides the time horizon into a number of nonoverlapping intervals.
- T_{kl} be the interval that starts in the period k and ends in period l. During T_{kl}, we can process any job j that has been released ($r_j \leq k$) and its deadline has not yet been reached ($l \leq d_j$).

8.3.1 Example

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- 4 jobs with release times 3, 1, 3, 5, and deadlines 5, 4, 7, 9.
- The ascending list of release times and deadlines is $1, 3, 4, 5, 7, 9$.
- Five intervals: T_{13}, T_{34}, T_{45}, T_{57}, T_{79}.

8.3.2 Network

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- Nodes: source s, sink t, a node corresponding to each job j, and a node corresponding to each interval T_{kl}.
- Arcs: (s, j), with capacity p_j. Flow represents the number of periods of processing that job j receives.
- Arcs: (T_{kl}, t), with capacity $m(l - k)$. Flow represents the total number of machine-periods of processing during T_{kl}.

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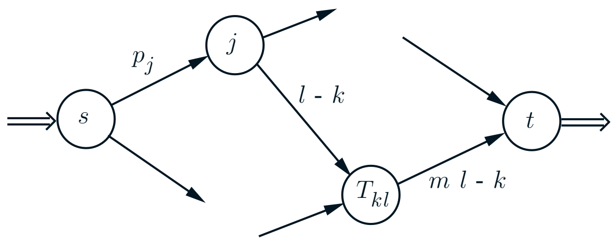
- Arcs: (j, T_{kl}) if $r_j \leq k \leq l \leq d_j$ with capacity $l - k$. Flow represents the number of periods that job j is processed during T_{kl}.



9 Min-Cost Flow

9.1 Passenger Routing

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- United Airlines has seven daily flights from BOS to SFO, every two hours, starting at 7am.
- Capacities are 100, 100, 100, 150, 150, 150, and ∞.
- Passengers suffering from overbooking are diverted to later flights.
- Delayed passengers get \$200 plus \$20 for every hour of delay.
- Suppose that today the first six flighs have 110, 160, 103, 149, 175, and 140 confirmed reservations.

Determine the most economical passenger routing strategy!

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Fall 2009