15.081J/6.251J Introduction to Mathematical Programming

Lecture 8: Duality Theory I

1 Outline

SLIDE 1

- Motivation of duality
- General form of the dual
- Weak and strong duality
- Relations between primal and dual
- Economic Interpretation
- Complementary Slackness

2 Motivation

2.1 An idea from Lagrange

SLIDE 2

Consider the LOP, called the **primal** with optimal solution $\boldsymbol{x}^*$

$$\begin{array}{ll} \min & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

Relax the constraint

$$\begin{array}{rl} g(\boldsymbol{p}) = \min & \boldsymbol{c}'\boldsymbol{x} + \boldsymbol{p}'(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}) \\ \text{s.t.} & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

$$g(\boldsymbol{p}) \leq \boldsymbol{c}'\boldsymbol{x}^* + \boldsymbol{p}'(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}^*) = \boldsymbol{c}'\boldsymbol{x}^*$$

Get the tightest lower bound, i.e.,

$$\max g(p)$$

$$\begin{aligned} g(\boldsymbol{p}) &= \min_{\boldsymbol{x} \geq \boldsymbol{0}} \left[\boldsymbol{c}'\boldsymbol{x} + \boldsymbol{p}'(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}) \right] \\ &= \boldsymbol{p}'\boldsymbol{b} + \min_{\boldsymbol{x} \geq \boldsymbol{0}} (\boldsymbol{c}' - \boldsymbol{p}'\boldsymbol{A})\boldsymbol{x} \end{aligned}$$

Note that

$$\min_{\boldsymbol{x} \geq \boldsymbol{0}} (\boldsymbol{c}' - \boldsymbol{p}'\boldsymbol{A})\boldsymbol{x} = \begin{cases} 0, & \text{if } \boldsymbol{c}' - \boldsymbol{p}'\boldsymbol{A} \geq \boldsymbol{0}', \\ -\infty, & \text{otherwise.} \end{cases}$$

$$\textbf{Dual} \quad \max \ g(\boldsymbol{p}) \quad \Leftrightarrow \quad \begin{array}{ll} \max & \boldsymbol{p}'\boldsymbol{b} \\ \text{s.t.} & \boldsymbol{p}'\boldsymbol{A} \leq \boldsymbol{c}' \end{array}$$

3 General form of the dual

SLIDE 3

$$
\begin{array}{llll|llll}
\textbf{Primal} & & & & \textbf{Dual} & & \\
\min & \mathbf{c}'\mathbf{x} & & & \max & \mathbf{p}'\mathbf{b} & \\
\text{s.t.} & \mathbf{a}_i'\mathbf{x} \geq b_i & i \in M_1 & & \text{s.t.} & p_i \geq 0 & i \in M_1 \\
 & \mathbf{a}_i'\mathbf{x} \leq b_i & i \in M_2 & & & p_i \leq 0 & i \in M_1 \\
 & \mathbf{a}_i'\mathbf{x} = b_i & i \in M_3 & & & p_i \ {}^{>}_{<} 0 & i \in M_3 \\
 & x_j \geq 0 & j \in N_1 & & & \mathbf{p}'\mathbf{A}_j \leq c_j & j \in N_1 \\
 & x_j \leq 0 & j \in N_2 & & & \mathbf{p}'\mathbf{A}_j \geq c_j & j \in N_2 \\
 & x_j \ {}^{>}_{<} 0 & j \in N_3 & & & \mathbf{p}'\mathbf{A}_j = c_j & j \in N_3
\end{array}
$$

3.1 Example

SLIDE 4

$$
\begin{array}{ll}
\min & x_1 + 2x_2 + 3x_3 \\
\text{s.t.} & -x_1 + 3x_2 = 5 \\
 & 2x_1 - x_2 + 3x_3 \geq 6 \\
 & x_3 \leq 4 \\
 & x_1 \geq 0 \\
 & x_2 \leq 0 \\
 & x_3 \text{ free,}
\end{array}
\qquad
\begin{array}{ll}
\max & 5p_1 + 6p_2 + 4p_3 \\
\text{s.t.} & p_1 \text{ free} \\
 & p_2 \geq 0 \\
 & p_3 \leq 0 \\
 & -p_1 + 2p_2 \leq 1 \\
 & 3p_1 - p_2 \geq 2 \\
 & 3p_2 + p_3 = 3.
\end{array}
$$

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Primal	min	max	**dual**
constraints	$\geq b_i$ $\leq b_i$ $= b_i$	≥ 0 ≤ 0 ${}^{>}_{<}0$	variables
variables	≥ 0 ≤ 0 ${}^{>}_{<}0$	$\leq c_j$ $\geq c_j$ $= c_j$	constraints

Theorem: The dual of the dual is the primal.

3.2 A matrix view

SLIDE 6

$$
\begin{array}{ll}
\min & \mathbf{c}'\mathbf{x} \\
\text{s.t.} & \mathbf{A}\mathbf{x} = \mathbf{b} \\
 & \mathbf{x} \geq \mathbf{0}
\end{array}
\qquad
\begin{array}{ll}
\max & \mathbf{p}'\mathbf{b} \\
\text{s.t.} & \mathbf{p}'\mathbf{A} \leq \mathbf{c}'
\end{array}
$$

$$
\begin{array}{ll}
\min & \mathbf{c}'\mathbf{x} \\
\text{s.t.} & \mathbf{A}\mathbf{x} \geq \mathbf{b}
\end{array}
\qquad
\begin{array}{ll}
\max & \mathbf{p}'\mathbf{b} \\
\text{s.t.} & \mathbf{p}'\mathbf{A} = \mathbf{c}' \\
 & \mathbf{p} \geq \mathbf{0}
\end{array}
$$

4 Weak Duality

SLIDE 7

Theorem:
If $\mathbf{x}$ is primal feasible and $\mathbf{p}$ is dual feasible then $\mathbf{p}'\mathbf{b} \leq \mathbf{c}'\mathbf{x}$
Proof
$\mathbf{p}'\mathbf{b} = \mathbf{p}'\mathbf{A}\mathbf{x} \leq \mathbf{c}'\mathbf{x}$

Corollary:
If $\boldsymbol{x}$ is primal feasible, $\boldsymbol{p}$ is dual feasible, and $\boldsymbol{p}'\boldsymbol{b} = \boldsymbol{c}'\boldsymbol{x}$, then $\boldsymbol{x}$ is optimal in the primal and $\boldsymbol{p}$ is optimal in the dual.

5 Strong Duality

SLIDE 8

Theorem: If the LOP has optimal solution, then so does the dual, and optimal costs are equal.
Proof:

$$\begin{array}{rrcl} \min & \boldsymbol{c}'\boldsymbol{x} & & \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} & = & \boldsymbol{b} \\ & \boldsymbol{x} & \geq & \boldsymbol{0} \end{array}$$

Apply Simplex; optimal solution $\boldsymbol{x}$, basis $\boldsymbol{B}$.
Optimality conditions:

$$\boldsymbol{c}' - \boldsymbol{c}_B'\boldsymbol{B}^{-1}\boldsymbol{A} \geq \boldsymbol{0}'$$

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Define $\boldsymbol{p}' = \boldsymbol{c}_B'\boldsymbol{B}^{-1} \Rightarrow \boldsymbol{p}'\boldsymbol{A} \leq \boldsymbol{c}'$
$\Rightarrow \boldsymbol{p}$ dual feasible for

$$\begin{array}{rl} \max & \boldsymbol{p}'\boldsymbol{b} \\ \text{s.t.} & \boldsymbol{p}'\boldsymbol{A} \leq \boldsymbol{c}' \end{array}$$

$$\boldsymbol{p}'\boldsymbol{b} = \boldsymbol{c}_B'\boldsymbol{B}^{-1}\boldsymbol{b} = \boldsymbol{c}_B'\boldsymbol{x}_B = \boldsymbol{c}'\boldsymbol{x}$$

$\Rightarrow \boldsymbol{x}, \boldsymbol{p}$ are primal and dual optimal

5.1 Intuition

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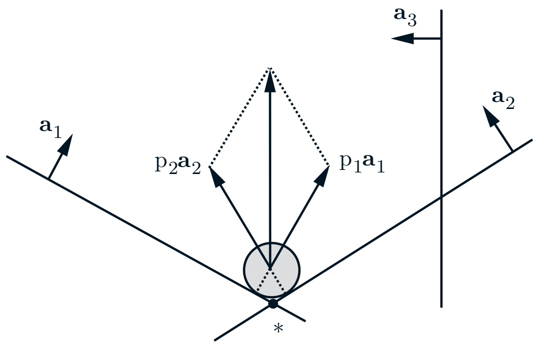

6 Relations between primal and dual

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	Finite opt.	Unbounded	Infeasible
Finite opt.	*		
Unbounded			*
Infeasible		*	*

7 Economic Interpretation

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- $\boldsymbol{x}$ optimal nondegenerate solution: $\boldsymbol{B}^{-1}\boldsymbol{b} > \boldsymbol{0}$
- Suppose $\boldsymbol{b}$ changes to $\boldsymbol{b} + \boldsymbol{d}$ for some small $\boldsymbol{d}$
- How is the optimal cost affected?
- For small $\boldsymbol{d}$ feasibilty unaffected
- Optimality conditions unaffected
- New cost $\quad \boldsymbol{c}'_B\boldsymbol{B}^{-1}(\boldsymbol{b}+\boldsymbol{d}) = \boldsymbol{p}'(\boldsymbol{b}+\boldsymbol{d})$
- If resource i changes by d_i, cost changes by $p_i d_i$: "Marginal Price"

8 Complementary slackness

8.1 Theorem

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Let $\boldsymbol{x}$ primal feasible and $\boldsymbol{p}$ dual feasible. Then $\boldsymbol{x}, \boldsymbol{p}$ optimal **if and only if**

$$p_i(\boldsymbol{a}'_i\boldsymbol{x} - b_i) = 0, \quad \forall i$$

$$x_j(c_j - \boldsymbol{p}'\boldsymbol{A}_j) = 0, \quad \forall j$$

8.2 Proof

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- $u_i = p_i(\boldsymbol{a}'_i\boldsymbol{x} - b_i)$ and $v_j = (c_j - \boldsymbol{p}'\boldsymbol{A}_j)x_j$
- If $\boldsymbol{x}$ primal feasible and $\boldsymbol{p}$ dual feasible, we have $u_i \geq 0$ and $v_j \geq 0$ for all i and j.
- Also
$$\boldsymbol{c}'\boldsymbol{x} - \boldsymbol{p}'\boldsymbol{b} = \sum_i u_i + \sum_j v_j.$$
- By the strong duality theorem, if $\boldsymbol{x}$ and $\boldsymbol{p}$ are optimal, then $\boldsymbol{c}'\boldsymbol{x} = \boldsymbol{p}'\boldsymbol{b} \Rightarrow$ $u_i = v_j = 0$ for all i, j.
- Conversely, if $u_i = v_j = 0$ for all i, j, then $\boldsymbol{c}'\boldsymbol{x} = \boldsymbol{p}'\boldsymbol{b}$,
- $\Rightarrow \boldsymbol{x}$ and $\boldsymbol{p}$ are optimal.

8.3 Example

SLIDE 15

$$
\begin{array}{rrcrcrcl}
\min & 13x_1 & + & 10x_2 & + & 6x_3 & & \\
\text{s.t.} & 5x_1 & + & x_2 & + & 3x_3 & = & 8 \\
& 3x_1 & + & x_2 & & & = & 3 \\
& x_1 & , & x_2 & , & x_3 & \geq & 0
\end{array}
\qquad
\begin{array}{rrcrcl}
\max & 8p_1 & + & 3p_2 & & \\
\text{s.t.} & 5p_1 & + & 3p_2 & \leq & 13 \\
& p_1 & + & p_2 & \leq & 10 \\
& 3p_1 & & & \leq & 6
\end{array}
$$

Is $\boldsymbol{x}^* = (1, 0, 1)'$ optimal?

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$$5p_1 + 3p_2 = 13, \quad 3p_1 = 6$$

$$\Rightarrow p_1 = 2, \quad p_2 = 1$$

Objective=19