15.081J/6.251J Introduction to Mathematical Programming

Lecture 2: Geometry of Linear Optimization I

1 Outline:

SLIDE 1

1. What is the central problem?
2. Standard Form.
3. Preliminary Geometric Insights.
4. Geometric Concepts (Polyhedra, "Corners").
5. Equivalence of algebraic and geometric concepts.

2 Central Problem

SLIDE 2

$$
\begin{array}{lll}
\text{minimize} & \boldsymbol{c}'\boldsymbol{x} & \\
\text{subject to} & \boldsymbol{a_i}'\boldsymbol{x} = b_i & i \in M_1 \\
 & \boldsymbol{a_i}'\boldsymbol{x} \le b_i & i \in M_2 \\
 & \boldsymbol{a_i}'\boldsymbol{x} \ge b_i & i \in M_3 \\
 & x_j \ge 0 & j \in N_1 \\
 & x_j \gtrless 0 & j \in N_2
\end{array}
$$

2.1 Standard Form

SLIDE 3

$$
\begin{array}{ll}
\text{minimize} & \boldsymbol{c}'\boldsymbol{x} \\
\text{subject to} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\
 & \boldsymbol{x} \ge \boldsymbol{0}
\end{array}
$$

Characteristics

- Minimization problem
- Equality constraints
- Non-negative variables

2.2 Transformations

SLIDE 4

$$
\begin{array}{lcl}
\max \boldsymbol{c}'\boldsymbol{x} & & -\min(-\boldsymbol{c}'\boldsymbol{x}) \\
\boldsymbol{a_i}'\boldsymbol{x} \le b_i & & \boldsymbol{a_i}'\boldsymbol{x} + s_i = b_i, \quad s_i \ge 0 \\
 & \Leftrightarrow & \\
\boldsymbol{a_i}'\boldsymbol{x} \ge b_i & & \boldsymbol{a_i}'\boldsymbol{x} - s_i = b_i, \quad s_i \ge 0 \\
x_j \gtrless 0 & & x_j = x_j^+ - x_j^- \\
 & & x_j^+ \ge 0, \; x_j^- \ge 0
\end{array}
$$

2.3 Example

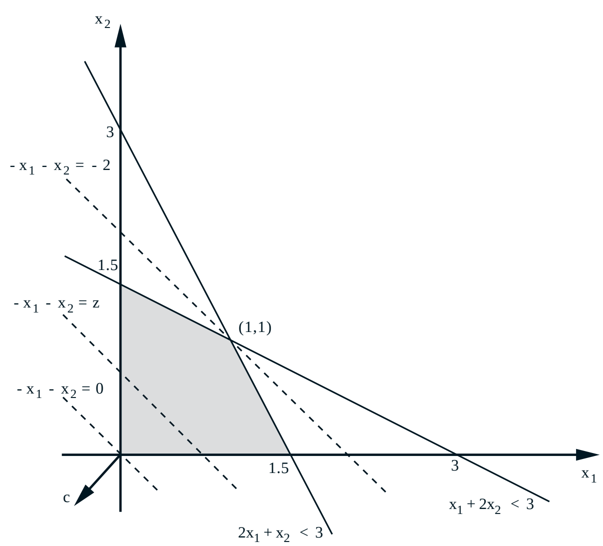
SLIDE 5

$$
\begin{array}{ll}
\text{maximize} & x_1 - x_2 \\
\text{subject to} & x_1 + x_2 \leq 1 \\
 & x_1 + 2x_2 \geq 1 \\
 & x_1 \, {}^{>}_{<} 0, x_2 \geq 0 \\
 & \Downarrow \\
-\text{minimize} & -x_1^+ + x_1^- + x_2 \\
\text{subject to} & x_1^+ - x_1^- + x_2 + s_1 \qquad = 1 \\
 & x_1^+ - x_1^- + 2x_2 \qquad - s_2 = 1 \\
 & x_1^+, x_1^-, x_2, s_1, s_2 \geq 0
\end{array}
$$

3 Preliminary Insights

SLIDE 6

$$
\begin{array}{rl}
\text{minimize} & -x_1 - x_2 \\
\text{subject to} & x_1 + 2x_2 \leq 3 \\
 & 2x_1 + x_2 \leq 3 \\
 & x_1, x_2 \geq 0
\end{array}
$$

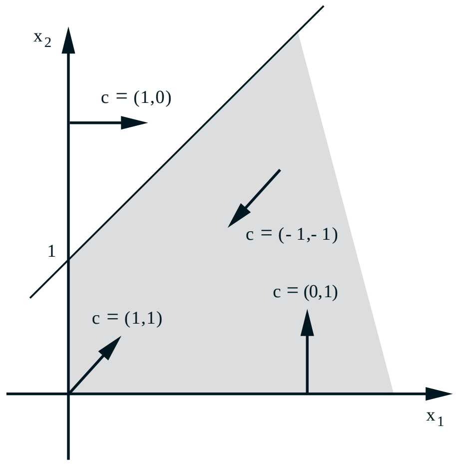


SLIDE 7

$$
\begin{array}{rcl}
-x_1 + x_2 & \leq & 1 \\
x_1 & \geq & 0 \\
x_2 & \geq & 0
\end{array}
$$

SLIDE 8

- There exists a unique optimal solution.
- There exist multiple optimal solutions; in this case, the set of optimal solutions can be either bounded or unbounded.



- The optimal cost is $-\infty$, and no feasible solution is optimal.
- The feasible set is empty.

4 Polyhedra

4.1 Definitions

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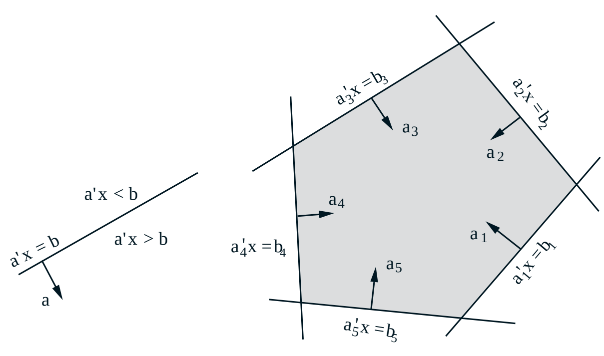
- The set $\{\boldsymbol{x} \mid \boldsymbol{a}'\boldsymbol{x} = b\}$ is called a **hyperplane**.
- The set $\{\boldsymbol{x} \mid \boldsymbol{a}'\boldsymbol{x} \geq b\}$ is called a **halfspace**.
- The intersection of many halfspaces is called a **polyhedron**.
- A polyhedron P is a convex set, i.e., if $\boldsymbol{x}, \boldsymbol{y} \in P$, then $\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y} \in P$.



(a) (b)

5 Corners

5.1 Extreme Points

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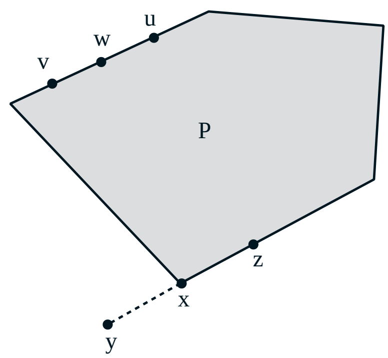
- Polyhedron $P = \{\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}\}$
- $\boldsymbol{x} \in P$ is **an extreme point** of P

  if $\nexists\, \boldsymbol{y}, \boldsymbol{z} \in P$ $(\boldsymbol{y} \neq \boldsymbol{x}, \boldsymbol{z} \neq \boldsymbol{x})$:

  $\boldsymbol{x} = \lambda \boldsymbol{y} + (1-\lambda)\boldsymbol{z}, 0 < \lambda < 1$



5.2 Vertex

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- $\boldsymbol{x} \in P$ is **a vertex** of P if $\exists\, \boldsymbol{c}$:

  $\boldsymbol{x}$ is the unique optimum

$$
\begin{array}{ll}
\text{minimize} & \boldsymbol{c}'\boldsymbol{y} \\
\text{subject to} & \boldsymbol{y} \in P
\end{array}
$$

5.3 Basic Feasible Solution

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$$P = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 1, \quad x_1, x_2, x_3 \geq 0\}$$

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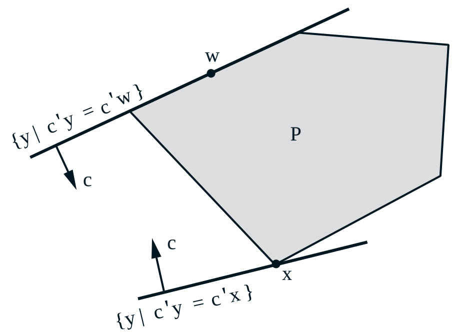
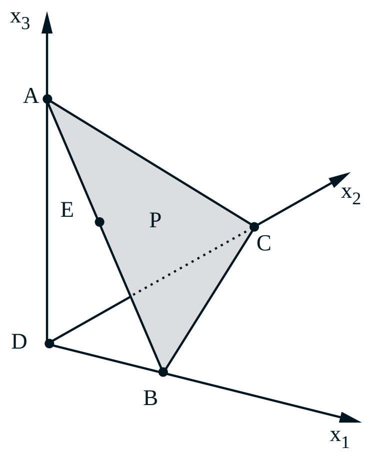
Points A,B,C : 3 constraints active

Point E: 2 constraints active

suppose we add $2x_1 + 2x_2 + 2x_3 = 2$.





Then 3 hyperplanes are tight, but constraints are not linearly independent.

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Intuition: a point at which n inequalities are tight and corresponding equations are linearly independent.
$P = \{\boldsymbol{x} \in \Re^n \mid \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}$

- $\boldsymbol{a_1}, \ldots, \boldsymbol{a_m}$ rows of $\boldsymbol{A}$
- $\boldsymbol{x} \in P$
- $I = \{i \mid \boldsymbol{a_i}'\boldsymbol{x} = b_i\}$

Definition $\boldsymbol{x}$ is **a basic feasible solution** if subspace spanned by $\{\boldsymbol{a_i}, i \in I\}$ is $\Re^n$.

5.3.1 Degeneracy

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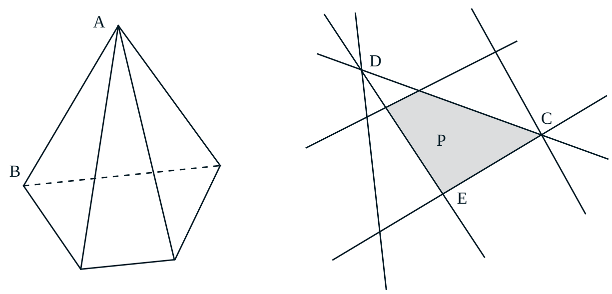
- If $|I| = n$, then $\boldsymbol{a_i},\ i \in I$ are linearly independent; $\boldsymbol{x}$ nondegenerate.

- If $|I| > n$, then there exist n linearly independent $\{\boldsymbol{a_i}, i \in I\}$; $\boldsymbol{x}$ degenerate.



(a) (b)

5.3.2 Example

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$$\begin{array}{llll l}
\min & x_1 + & 5x_2 & -2x_3 & \\
\text{s.t.} & x_1 + & x_2 + & x_3 & \leq 4 \\
 & x_1 & & & \leq 2 \\
 & & & x_3 & \leq 3 \\
 & & 3x_2 + & x_3 & \leq 6 \\
 & x_1, & x_2, & x_3 & \geq 0
\end{array}$$

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6 Equivalence of definitions

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Theorem: $P = \{\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}$. Let $\boldsymbol{x} \in P$.
$\boldsymbol{x}$ is a vertex $\Leftrightarrow$ $\boldsymbol{x}$ is an extreme point $\Leftrightarrow$ $\boldsymbol{x}$ is a BFS.

6.1 Proof

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1. **Vertex $\Rightarrow$ extreme point**

   $\exists \boldsymbol{c} : \boldsymbol{c}'\boldsymbol{x} < \boldsymbol{c}'\boldsymbol{y} \quad \forall \boldsymbol{y} \in P$

   If $\boldsymbol{x}$ is not an extreme point $\exists \boldsymbol{y}, \boldsymbol{z} \neq \boldsymbol{x}$:

   $\boldsymbol{x} = \lambda \boldsymbol{y} + (1-\lambda)\boldsymbol{z}$. But $\boldsymbol{c}'\boldsymbol{x} < \boldsymbol{c}'\boldsymbol{y}, \quad \boldsymbol{c}'\boldsymbol{x} < \boldsymbol{c}'\boldsymbol{z}$

   $\Rightarrow \boldsymbol{c}'\boldsymbol{x} = \lambda \boldsymbol{c}'\boldsymbol{y} + (1-\lambda)\boldsymbol{c}'\boldsymbol{z} < \boldsymbol{c}'\boldsymbol{x}$ **contradiction**

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2. **Extreme point $\Rightarrow$ BFS**

   Suppose $\boldsymbol{x}$ is not a BFS.

Let $I = \{i : \boldsymbol{a_i'x} = b_i\}$. But $\boldsymbol{a_i}$ do not

span all of $\Re^n \Rightarrow \exists \boldsymbol{z} \in \Re^n : \boldsymbol{a_i'z} = \boldsymbol{0}, i \in I$

Let

$\boldsymbol{x_1} = \boldsymbol{x} + \epsilon \boldsymbol{z}$

$\boldsymbol{x_2} = \boldsymbol{x} - \epsilon \boldsymbol{z}$

$$\left.\begin{array}{l} \boldsymbol{a_i'x_1} = b_i \\ \boldsymbol{a_i'x_2} = b_i \end{array}\right\} i \in I$$

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$i \notin I : \boldsymbol{a_i'x} < b_i \Rightarrow \boldsymbol{a_i}'(\boldsymbol{x} + \epsilon\boldsymbol{z}) < b_i, \quad \boldsymbol{a_i}'(\boldsymbol{x} - \epsilon\boldsymbol{z}) < b_i$
for ϵ small enough.
$\Rightarrow \boldsymbol{x_1}, \boldsymbol{x_2} \in P$: yet $\boldsymbol{x} = \frac{\boldsymbol{x_1}+\boldsymbol{x_2}}{2} \Rightarrow$
$\boldsymbol{x}$ not an extreme point: **contradiction**

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3. BFS $\Rightarrow$ vertex

$\boldsymbol{x}^*$ BFS
$I = \{i : \boldsymbol{a_i'x}^* = b_i\}$
Let $d_i = \begin{cases} 1 & i \in I \\ 0 & i \notin I. \end{cases}$
$\boldsymbol{c}' = -\boldsymbol{d'A}$

Then $\boldsymbol{c'x}^* = -\boldsymbol{d'Ax}^* = -\sum_{i=1}^{m} d_i \boldsymbol{a_i'x}^* = -\sum_{i \in I} \boldsymbol{a_i'x}^* = -\sum_{i \in I} b_i.$

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But $\forall \boldsymbol{x} \in \boldsymbol{P} : \boldsymbol{a_i'x} \le b_i \Rightarrow$

$\boldsymbol{c'x} = -\sum_{i \in I} \boldsymbol{a_i'x} \ge -\sum_{i \in I} b_i = \boldsymbol{c'x}^*$

$\boldsymbol{x}^*$ optimum

$$\begin{array}{ll} \min & \boldsymbol{c'x} \\ & \boldsymbol{x} \in P. \end{array}$$

Why unique?

Equality holds if $\boldsymbol{a_i'x} = b_i, i \in I$; since $\boldsymbol{a_i}$ spans $\Re^n$, $\boldsymbol{a_i'x} = b_i$ has a unique solution $\boldsymbol{x} = \boldsymbol{x}^*$.

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