

Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science
6.245: MULTIVARIABLE CONTROL SYSTEMS

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Algorithms for H-Infinity Optimization¹

This lecture covers the algorithmic side of H-Infinity optimization. It presents classical necessary and sufficient conditions of existence of a γ -suboptimal controller, stated in terms of stabilizing solutions of Riccati equations, and provides explicit formulae for the so-called “central” γ -suboptimal controller.

7.1 Problem Formulation and Algorithm Objectives

This section examines the specifics of the way in which the H-Infinity optimal LTI feedback design problem is formulated (and, eventually, solved).

7.1.1 Suboptimal Feedback Design

H-infinity optimization problem appears to be formulated as the task of designing a stabilizing controller $K(s)$, which minimizes the H-infinity norm of the closed-loop transfer matrix T_{zw} from w to z for a given open loop plant $P(s)$ (see Figure 7.1), defined by state space equations

$$\begin{aligned}\dot{x} &= Ax + B_1w + B_2u \\ z &= C_1x + D_{11}w + D_{12}u \\ y &= C_2x + D_{21}w + D_{22}u\end{aligned}$$

A set of standard well posedness constraints is imposed on the setup:

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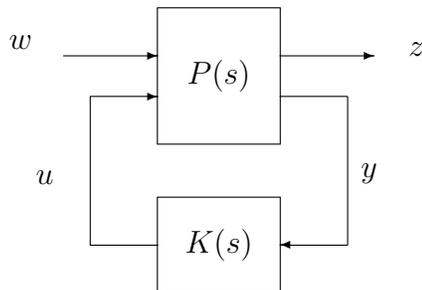


Figure 7.1: General LTI design setup

- for output feedback stabilizability, the pairs (A, B_2) and (C_2, A) must be respectively stabilizable and detectable
- for non-singularity, D_{21} must be right invertible (full measurement noise), D_{12} must be left invertible (full control penalty), and matrices

$$\begin{bmatrix} A - sI & B_2 \\ C_1 & D_{12} \end{bmatrix}, \quad \begin{bmatrix} A - sI & B_1 \\ C_2 & D_{21} \end{bmatrix}$$

must be respectively left and right invertible for all $s \in j\mathbf{R}$

However, in contrast with the case of H2 optimization, basic H-Infinity algorithms solve a *suboptimal* controller design problem, formulated as that of finding whether, for a given $\gamma > 0$, a controller achieving the closed loop L2 gain $\|T_{zw}\|_\infty < \gamma$ exists, and, in case the answer is affirmative, calculating one such controller.

7.1.2 Why Suboptimal Controllers?

There are several reasons to prefer suboptimal controllers over the optimal one in H-Infinity optimization. One of the most compelling reasons is that the optimal closed loop transfer matrix T_{zw} can be shown to have a *constant* largest singular number over the complete frequency range. In particular, this means that the optimal controller is not strictly proper, and the optimal frequency response to the cost output does not roll off at high frequencies.

Example 7.1 Consider the standard LTI feedback optimization setup with

$$P(s) = \begin{bmatrix} \frac{1}{s+1} & -1 \\ \frac{1-s}{1+s} & 0 \end{bmatrix},$$

(a case of state estimation). Since there is no feedback loop to close here, the controller transfer function $K(s)$ must be a stable, and the cost is the H-Infinity norm

$$\|T_{zw}\|_{\infty} \rightarrow \min, \quad T_{zw} = \frac{1-s}{1+s}K(s) - \frac{1}{1+s}.$$

Since $T_{zw}(1) = 0.5$ for all K , and $\|T\|_{\infty} \geq |T(1)|$ for all T , the norm $\|T_{zw}\|_{\infty}$ cannot be less than 0.5. Also, there is the only way to make it identically equal to 0.5, by using $K_0(s) \equiv 0.5$. Hence $K_0(s) = 0.5$ is the (only) H-Infinity optimal controller, and the optimal T_{zw} has a flat (at 0.5) amplitude of frequency response.

Note that the H2 optimal $K(s)$ equals zero in this case.

Another good reason is that, in a setup with more than one control variable or more than one sensor variable, the optimal H-Infinity controller is frequently not unique (indeed, there could be a continuum of H-Infinity optimal controllers). While some of those optimal controllers could be arguably better than the other, the “most optimal” ones tend to have an order much larger than the necessary minimum (which equals the order of the plant).

Example 7.2 Consider the standard LTI feedback optimization setup with 2 sensors, 2 controls, and with

$$P(s) = \begin{bmatrix} \frac{1}{s+1} & 0 & -1 & 0 \\ 0 & \frac{1}{2} \frac{1}{s+1} & 0 & -1 \\ \frac{1-s}{1+s} & 0 & 0 & 0 \\ 0 & \frac{1-s}{1+s} & 0 & 0 \end{bmatrix}$$

(this is essentially the optimization task from the previous example, repeated twice with some scaling). Due to the same arguments as before, the closed loop H-Infinity norm from w to z cannot be made smaller than 0.5. However, there are plenty of “controllers” $K(s)$ which will make the closed loop H-Infinity norm exactly equal to 0.5: for example, one can take

$$K(s) = \begin{bmatrix} 0.5 & 0 \\ 0 & k_2 \end{bmatrix},$$

which will be an optimal controller for every $k_2 \in [0, 0.5]$.

A somewhat less compelling reason for not using H-Infinity optimal controllers is that, while it is known how to find them using ideal algebraic calculations, implementing these formulae in an efficient numerical algorithm is a tough problem.

7.1.3 What is done by the software

The μ -Analysis and Synthesis Toolbox provides function `hinfsyn.m` for calculating a suboptimal output feedback:

`function [k,g,gfin]=hinfsyn(p,nmeas,ncon,gmin,gmax,tol)`

(some additional optional inputs and outputs are possible here). Here “p” is the packed model of $P(s)$, “ncon” and “nmeas” are the numbers of actuators and sensors, “gmin” and “gmax” are a lower and an upper a-priori bounds of the achievable closed-loop H-infinity norm. The program performs a binary search for the “level of optimality” parameter γ . At any point, it operates with a “current guess” γ of the minimal achievable closed loop H-infinity norm, a lower bound γ_- and an upper bound γ_+ of γ . The initial values of γ_- and γ_+ are supplied by “gmin” and “gmax”, and, initially, $\gamma = \gamma_+$. At each iteration of the algorithm, it is checked whether it is possible to design a controller with the resulting closed loop H-infinity gain less than γ . If the answer is positive, new values of γ and γ_+ are assigned, according to

$$\gamma^{new} = 0.5(\gamma_-^{old} + \gamma^{old}), \quad \gamma_+^{new} = \gamma^{old}.$$

If the answer is negative, new values of γ and γ_- are assigned, according to

$$\gamma^{new} = 0.5(\gamma_+^{old} + \gamma^{old}), \quad \gamma_-^{new} = \gamma^{old}.$$

This process is continued until the relative difference between γ_+ and γ_- becomes smaller than the tolerance parameter “tol”. Then, actual suboptimal feedback design is performed, with γ_+ being the target H-infinity performance.

For a given γ , in order to check that a γ -suboptimal solution exists, the algorithm calls for forming two auxiliary abstract H2 optimization problems. Roughly speaking, one of these equations corresponds to the limitations in the “full information feedback” part of the stabilization task, and the other corresponds to the “zero sensor input” part of the stabilization task. For feasibility of the suboptimal H-infinity control problem, stabilizing solutions X and Y of some Riccati equations must exist and be non-negative definite. In addition, a coupling condition $\lambda_{\max}(YX) < \gamma^2$ must be satisfied.

When checking solvability of the Riccati equations, “hinfsyn” uses the “Hamiltonian matrices approach”, which associates stabilizing (i.e. such that $\beta - \rho \alpha$ is a Hurwitz matrix) solutions $\alpha = \alpha'$ of an algebraic Riccati equation

$$\alpha + \alpha \beta + \beta' \alpha = \alpha \rho \alpha$$

with the stable invariant subspace of the auxiliary Hamiltonian matrix

$$H = \begin{bmatrix} \beta & \rho \\ \alpha & -\beta' \end{bmatrix}.$$

Unlike in the Riccati equations and Hamiltonian matrices associated with the usual KYP Lemma, matrix ρ will not be positive semidefinite. This will be somewhat compensated by validity of inequality $\alpha \geq 0$.

7.2 Background Results

This section contains general mathematical results which play an important role in understanding H-Infinity optimization. In addition to providing a specific version of the KYP Lemma and a statement on computing stabilizing solutions of general algebraic Riccati equation, it presents the *Parrot's theorem* and its generalizations.

7.2.1 A Specific Version of the KYP Lemma

In general, KYP Lemma does not provide information about sign definiteness of matrix $P = P'$ of coefficients of the associated quadratic storage function. However, there exists a number of situations when positivity of P is implied by the setup. To derive a solution of H-Infinity optimization problem, we need one such result.

Theorem 7.1 *The following conditions are equivalent:*

- (a) *a is a Hurwitz matrix, and $G(s) = d + c(sI - a)^{-1}b$ has H-Infinity norm less than γ ;*
- (b) *there exists $p = p' > 0$ such that the quadratic form*

$$\sigma_p(x, w) = \gamma^2|w|^2 - |cx + dw|^2 - 2x'p(ax + bw)$$

is strictly positive definite.

Proof Taking the KYP lemma into account, the only thing to prove here is that, subject to positive definiteness of σ_p , $p > 0$ whenever a is a Hurwitz matrix. Indeed, $\sigma_p \gg 0$ implies the Lyapunov inequality

$$pa + a'p + c'c < 0,$$

hence $p > 0$ if and only if a is a Hurwitz matrix. ■

7.2.2 Stabilizing Solutions of General Riccati Equations

To present a solution to the H-Infinity optimization problem, we need an important remark on uniqueness of stabilizing solutions \mathfrak{a} (i.e. such that $\beta - \rho\mathfrak{a}$ is a Hurwitz matrix) of the general Riccati equation

$$\alpha + \mathfrak{a}\beta + \beta'\mathfrak{a} = \mathfrak{a}\rho\mathfrak{a}, \tag{7.1}$$

where $\alpha = \alpha'$, β , $\rho = \rho'$ are given n -by- n matrices with real coefficients (ρ is not required to be sign definite), and \mathfrak{a} is an n -by- n real matrix to be found. Formally speaking, \mathfrak{a} is not required to be symmetric, but it will be shown that a stabilizing solution always is.

Theorem 7.2 *Riccati equation (7.1) has a stabilizing n -by- n solution \mathfrak{x} if and only if the maximal stable invariant subspace \mathcal{V}_+ of the associated Hamiltonian matrix*

$$H = \begin{bmatrix} \beta & \rho \\ \alpha & -\beta' \end{bmatrix} \quad (7.2)$$

has dimension n and contains no non-zero vectors in which the first n components are equal to zero. A stabilizing solution \mathfrak{x} (if exists) is unique, symmetric, and defined by

$$\mathfrak{x} = -[\psi_1 \ \psi_2 \ \dots \ \psi_n] \cdot [x_1 \ x_2 \ \dots \ x_n]^{-1}, \quad (7.3)$$

where

$$\begin{bmatrix} x_1 \\ \psi_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ \psi_2 \end{bmatrix}, \begin{bmatrix} x_n \\ \psi_n \end{bmatrix}$$

is an arbitrary basis in \mathcal{V}_+ .

Thus, a given algebraic Riccati equation cannot have more than one stabilizing solution. Moreover, checking existence and finding the solution, when it exists, is a relatively straightforward linear algebra operation.

Proof First, let us note that columns of a matrix V span an invariant subspace of a square matrix M if and only if $MV = VS$ for some square matrix S . Moreover, this subspace is stable if and only if S is a Hurwitz matrix, and anti-stable if and only if $-S$ is a Hurwitz matrix.

The proof follows easily from the observation that (7.1) is equivalent to

$$\begin{bmatrix} \beta & \rho \\ \alpha & -\beta' \end{bmatrix} \begin{bmatrix} I \\ -\mathfrak{x} \end{bmatrix} = \begin{bmatrix} I \\ -\mathfrak{x} \end{bmatrix} (\beta - \rho\mathfrak{x}),$$

as well as to

$$\begin{bmatrix} \beta' & \alpha \\ \rho & -\beta \end{bmatrix} \begin{bmatrix} \mathfrak{x} \\ I \end{bmatrix} = \begin{bmatrix} \mathfrak{x} \\ I \end{bmatrix} (\rho\mathfrak{x} - \beta).$$

Hence, if \mathfrak{x} is a stabilizing solution of (7.1) then the columns of $[I; -\mathfrak{x}]$ form a basis in a stable invariant subspace of \mathcal{H} , and the columns of $[\mathfrak{x}; I]$ form a basis in an anti-stable invariant subspace of \mathcal{H}' . Since the dimension of the maximal stable or anti-stable invariant subspace does not change with transposition of the matrix, the $2n$ -by- $2n$ matrix \mathcal{H} has an n -dimensional stable, and an n -dimensional anti-stable subspaces, which means that n is actually the dimension of the maximal stable invariant subspace of \mathcal{H} , and the columns of $[I; -\mathfrak{x}]$ form a basis in this subspace.

Conversely, if \mathcal{V}_+ is the n -dimensional stable invariant subspace of \mathcal{H} , the columns of $V = [F; G]$ form a basis in \mathcal{V}_+ , and $\det(F) \neq 0$ then

$$\begin{bmatrix} \beta & \rho \\ \alpha & -\beta' \end{bmatrix} \begin{bmatrix} F \\ G \end{bmatrix} = \begin{bmatrix} F \\ G \end{bmatrix} S$$

for some n -by- n Hurwitz matrix S . Hence (7.1) holds for $\mathfrak{x} = -GF^{-1}$, and $S = \beta - \rho\mathfrak{x}$.
 ■

7.2.3 Riccati Equalities and Inequalities

This section presents two frequently used relations between solutions of Riccati equations and Riccati inequalities.

The first relation is a straightforward implication of the KYP lemma.

Theorem 7.3 *Let $\rho = \rho' \geq 0$, $\alpha = \alpha'$ and β be n -by- n matrices with real coefficients, such that the pair (β, ρ) is stabilizable. Then the following conditions are equivalent:*

- (a) *the algebraic Riccati equation (7.1) has a stabilizing solution $\mathfrak{x} = \mathfrak{x}'$;*
- (b) *the strict Riccati inequality*

$$\alpha + \mathfrak{x}_0\beta + \beta'\mathfrak{x}_0 > \mathfrak{x}_0\rho\mathfrak{x}_0$$

has a solution $\mathfrak{x}_0 = \mathfrak{x}'_0$.

Moreover, $\mathfrak{x} > \mathfrak{x}_0$ whenever (a) and (b) are true.

Proof Let $\rho = ff'$. A stabilizing solution \mathfrak{x} from (a) defines a completion of squares relation

$$x'\alpha x + |u|^2 + 2x'\mathfrak{x}(\beta x + fu) = |u + f'\mathfrak{x}x|^2,$$

where $\beta - ff'\mathfrak{x}$ is a Hurwitz matrix. A solution \mathfrak{x}_0 from (b) means strict positive definiteness of the quadratic form

$$\sigma_0(x, u) = x'\alpha x + |u|^2 + 2x'\mathfrak{x}_0(\beta x + fu).$$

According to the KYP lemma, the two conditions are equivalent.

When both (a) and (b) hold, substituting $u = -f'\mathfrak{x}x$ and comparing the coefficient matrices of the resulting quadratic forms with respect to x yields

$$(\mathfrak{x}_0 - \mathfrak{x})(\beta - ff'\mathfrak{x}) + (\beta - ff'\mathfrak{x})'(\mathfrak{x}_0 - \mathfrak{x}) > 0,$$

which implies $\mathfrak{x}_0 - \mathfrak{x} < 0$, since $\beta - ff'\mathfrak{x}$ is a Hurwitz matrix.
 ■

The derivation of H-Infinity optimal control results relies on a less straightforward relation between Riccati equalities and inequalities.

Theorem 7.4 *Let $\alpha = \alpha' \geq 0$, $\rho = \rho'$, and β be n -by- n matrices with real coefficients, such that the pair (α, β) does not have unobservable modes on the imaginary axis (i.e. matrix $[\alpha; \beta - j\omega I]$ is left invertible for all $\omega \in \mathbf{R}$). Then the following conditions are equivalent:*

- (a) *the algebraic Riccati equation (7.1) has a stabilizing solution $\alpha = \alpha' \geq 0$;*
- (b) *the strict Riccati inequality*

$$\alpha + \alpha_0\beta + \beta'\alpha_0 < \alpha_0\rho\alpha_0$$

has a solution $\alpha_0 = \alpha'_0 > 0$.

Moreover, $\alpha < \alpha_0$ whenever (a) and (b) are true, and, for a given α in (a), the difference $\alpha_0 - \alpha$ can be made arbitrarily small by selecting an appropriate solution α_0 in (b).

Proof To show that (a) implies (b), note that, since $\beta - \rho\alpha$ is a Hurwitz matrix, equation

$$\Delta(\beta - \rho\alpha) + (\beta - \rho\alpha)'\Delta = -I$$

has a (unique) solution $\Delta = \Delta' > 0$. For $\alpha_0 = \alpha + \tau\Delta$ we have

$$\alpha + \alpha_0\beta + \beta'\alpha_0 - \alpha_0\rho\alpha_0 = -\tau I - \tau^2\Delta\rho\Delta,$$

which is a negative definite matrix for all sufficiently small $\tau > 0$.

To prove that (b) implies (a), note that multiplying the inequality by $p^0 = \alpha_0^{-1}$ on both side yields

$$\rho + p^0(-\beta') + (-\beta')'p^0 > p^0\alpha p^0. \quad (7.4)$$

When the pair $(-\beta', \alpha)$ is stabilizable, this inequality, according to Theorem 7.3, yields existence of a stabilizing solution $p = p' > p^0$ of

$$\rho + p(-\beta') + (-\beta')'p = p\alpha p.$$

Multiplying this equality by $\alpha = p^{-1}$ shows that α is a solution of (7.1). Since we also have

$$p(-\beta' - \alpha p)p^{-1} = \beta - \rho\alpha,$$

and $-\beta' - \alpha p$ is a Hurwitz matrix, $\beta - \rho\alpha$ is also a Hurwitz matrix, i.e. α is a stabilizing solution of (7.1).

In the case when the pair $(-\beta', \alpha)$ is not stabilizable, consider the block decomposition

$$\beta = \begin{bmatrix} \beta_{11} & \beta_{12} \\ 0 & \beta_{22} \end{bmatrix}, \quad \alpha = \begin{bmatrix} 0 & 0 \\ 0\alpha_{22} & \end{bmatrix},$$

where the pair $(-\beta'_{22}, \alpha_{22})$ is stabilizable, and β_{11} has no eigenvalues with positive real part. Since it was assumed that the pair (α, β) does not have unobservable modes on the imaginary axis, β_{11} has no eigenvalues on the imaginary axis. Considering the compatible block decompositions

$$p^0 = \begin{bmatrix} p_{11}^0 & p_{12}^0 \\ p_{21}^0 & p_{22}^0 \end{bmatrix}, \quad \rho = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix},$$

comparing the (2,2) blocks in (7.4) yields

$$\rho_{22} + p_{22}^0(-\beta'_{22}) + (-\beta'_{22})'p_{22}^0 > p_{22}^0\alpha_{22}p_{22}^0.$$

Since the pair $(-\beta'_{22}, \alpha_{22})$ is stabilizable, Theorem 7.3 yields existence of a stabilizing solution $p_{22} = p'_{22} > p_{22}^0$ of the Riccati equation

$$\rho_{22} + p_{22}(-\beta'_{22}) + (-\beta'_{22})'p_{22} = p_{22}\alpha_{22}p_{22}.$$

Hence $\mathfrak{x}_{22} = p_{22}^{-1}$ is a stabilizing solution of Riccati equation

$$\alpha_{22} + \mathfrak{x}_{22}\beta_{22} + \beta'_{22}\mathfrak{x}_{22} = \mathfrak{x}_{22}\rho_{22}\mathfrak{x}_{22},$$

and

$$\mathfrak{x} = \begin{bmatrix} 0 & 0 \\ 0 & \mathfrak{x}_{22} \end{bmatrix}$$

is a stabilizing solution of (7.1). ■

7.2.4 The Parrot's Lemma and Its Generalizations

The classical Parrot's lemma establishes an explicit solution formulae for the following matrix optimization question: given real matrices a, b, c of dimensions q -by- n , n -by- k , and m -by- n respectively, minimize

$$J(L) = \left\| \begin{bmatrix} a & b \\ c & L \end{bmatrix} \right\|$$

over the set of all m -by- k real matrices L .

Theorem 7.5 *The minimum of $J(L)$ equals*

$$\gamma = \max \left\{ \left\| \begin{bmatrix} a & b \end{bmatrix} \right\|, \left\| \begin{bmatrix} a \\ c \end{bmatrix} \right\| \right\},$$

and is achieved, in particular, with L defined (uniquely) by

$$Ly = \arg \max_v \inf_w \{ \gamma^2(|w|^2 + |y|^2) - |aw + by|^2 - |cw + v|^2 \}.$$

Theorem 7.5 is about defining a “control variable” v as a linear function $v = Lg$ of the “sensor variable” g , with an objective of satisfying a given quadratic inequality $\sigma(f, v, g) \geq 0$, where

$$\sigma(f, v, g) = \gamma^2(|f|^2 + |g|^2) - |af + bg|^2 - |cf + v|^2.$$

Since $g = 0$ would imply $v = Lg = 0$, the inequality $\sigma(f, 0, 0) \geq 0$ must be satisfied, which yields

$$\gamma \geq \left\| \begin{bmatrix} a \\ c \end{bmatrix} \right\|. \quad (7.5)$$

In addition, for every given pair (f, g) , the maximum of $\sigma(f, v, g)$ must be non-negative, which yields the inequality

$$\gamma \geq \left\| \begin{bmatrix} a & b \end{bmatrix} \right\|. \quad (7.6)$$

The non-trivial part of the Parrot’s theorem is concerned with establishing that

$$\sup_v \inf_f \sigma(f, v, g) \geq 0 \quad \forall g.$$

Since (7.6) means that

$$\inf_f \sup_v \sigma(f, v, g) \geq 0 \quad \forall g,$$

the Parrot’s lemma follows from the minimax statement

$$\inf_f \sup_v \sigma(f, v, g) = \sup_v \inf_f \sigma(f, v, g) \quad \forall g,$$

which in turn follows from the standard minimax theorem since $\sigma(f, v, g)$ is strictly concave with respect to v , and (7.5) means that $\sigma(f, v, g)$ is convex with respect to f .

In future derivations, the following generalization of the Parrot’s lemma will be used.

Theorem 7.6 *Let $\sigma = \sigma(f, v, g)$ be a quadratic form which is concave with respect to its second argument (i.e. $\sigma(0, v, 0) \geq 0$ for all v). Then the following conditions are equivalent:*

(a) there exists matrix L and $\epsilon > 0$ such that

$$\sigma(f, Lg, g) \geq \epsilon(|f|^2 + |g|^2) \quad \forall f, g;$$

(b) there exists $\epsilon > 0$ such that

$$\sigma(f, 0, 0) \geq \epsilon|f|^2, \quad \sup_v \sigma(f, v, g) \geq \epsilon(|f|^2 + |g|^2).$$

Moreover, when (b) is satisfied, L is defined by the condition that the quadratic form

$$\sigma_L(y) = \min_f \sigma(f, Ly, y)$$

is strictly positive definite.

Proof Following the arguments explaining the classical Parrot's theorem, it is easy to see that (a) implies (b). The fact that (b) implies (a) follows from the standard minimax theorem. ■

7.3 H-Infinity Optimization for a Simplified Setup

In this section, we present a derivation of necessary and sufficient conditions of existence of a γ -suboptimal feedback law, as well as a formula for one such feedback (the so-called *central controller*), for a slightly simplified version of the standard feedback design setup.

7.3.1 The Simplified Setup Solution

To keep the formulae simple, a commonly used *simplified setup* is considered here: noises do not enter the cost, no cross-term penalties between control and state variables are allowed, and sensor noises and plant disturbances are decoupled:

$$\dot{x} = Ax + \bar{B}_1 w_1 + B_2 u, \tag{7.7}$$

$$z = \begin{bmatrix} \bar{C}_1 x \\ u \end{bmatrix}, \tag{7.8}$$

$$y = C_2 x + w_2, \tag{7.9}$$

i.e.

$$B_1 = \begin{bmatrix} \bar{B}_1 & 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} \bar{C}_1 \\ 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 0 & I \end{bmatrix}.$$

The simplified setup is non-singular if and only if matrices

$$\begin{bmatrix} A - j\omega I & B_1 \end{bmatrix}, \begin{bmatrix} A' - j\omega I & C_1' \end{bmatrix} \quad (7.10)$$

are right invertible for all real ω . In addition, for stabilizing controllers to exist, the pair (A, B_2) must be stabilizable, and the pair (C_2, A) must be detectable.

Consider the following Riccati equations defined by the coefficients of (7.7)-(7.9) and a suboptimality level $\gamma > 0$:

$$\bar{C}_1' \bar{C}_1 + XA + A'X = X(B_2B_2' - \gamma^{-2}\bar{B}_1\bar{B}_1')X, \quad (7.11)$$

$$\bar{B}_1'\bar{B}_1 + YA' + AY = Y(C_2C_2' - \gamma^{-2}\bar{C}_1\bar{C}_1')Y, \quad (7.12)$$

with respect to symmetric matrices $X = X'$, $Y = Y'$.

Theorem 7.7 *Assume that the pair (A, B_2) is stabilizable, the pair (C_2, A) is detectable, and matrices in (7.10) are right invertible for all $\omega \in \mathbf{R}$. Then an LTI controller $K = K(s)$ such that the feedback $u = Ky$ stabilizes system (7.7)-(7.9) and yields a closed loop L_2 gain strictly less than $\gamma > 0$ exists if and only if Riccati equations (7.11),(7.12) have stabilizing solutions $X = X_\infty \geq 0$, $Y = Y_\infty \geq 0$, and*

$$\gamma^2 I > X_\infty^{1/2} Y_\infty X_\infty^{1/2}. \quad (7.13)$$

While the proof of Theorem 7.7 is constructive, and shows a simple way of calculating a suboptimal controller whenever it exists, the following more explicit statement is frequently used.

Theorem 7.8 *Under the conditions for the existence of a suboptimal H -Infinity controller from Theorem 7.7 are satisfied, an LTI controller $u = K(s)y$ is suboptimal if and only if it has a realization*

$$u(t) = -B_2' X_\infty \hat{x}(t) + v(t) \quad (7.14)$$

$$\dot{\hat{x}}(t) = A\hat{x}(t) + \bar{B}_1 \hat{w}_1(t) + B_2 u(t) + L_1(C_2 \hat{x}(t) - y(t)) + L_2 v(t), \quad (7.15)$$

$$v(t) = \Delta(s)(y(t) - C_2 \hat{x}(t)), \quad (7.16)$$

$$\hat{w}_1(t) = \gamma^{-2} \bar{B}_1' X_\infty \hat{x}(t), \quad (7.17)$$

where

$$L_1 = -(I - \gamma^{-2} Y_\infty X_\infty)^{-1} Y_\infty C_2', \quad L_2 = \gamma^{-2} Y_\infty X_\infty (I - \gamma^{-2} Y_\infty X_\infty)^{-1} B_2,$$

and $\Delta = \Delta(s)$ is a stable LTI system with H -Infinity norm strictly less than γ .

When $\Delta = 0$, Theorem 7.8 yields the so-called *central controller*

The proof of Theorem 7.7, given in the rest of this section, can be extended easily to the general non-simplified setup. In addition, the theorem can be strengthened by noting that using nonlinear controllers does not reduce the minimal achievable L2 gain. Also, a nice parameterization of *all* stabilizin LTI controllers which yield $\|T_{zw}\|_\infty < \gamma$ can be given in terms of X_∞, Y_∞ and the coefficients of (7.7)-(7.9).

7.3.2 Proof of Theorem 7.7

An LTI controller

$$\dot{x}_f = A_f x_f + B_f y, \quad u = C_f x_f + D_f y$$

defines a closed loop system

$$\frac{d\bar{x}}{dt} = a\bar{x} + bw, \quad z = c\bar{x} + dw,$$

with input w , output z , and state $\bar{x} = [x; x_f]$. According to Theorem 7.1, the controller is γ -suboptimal if and only if there exists a matrix $p = p' > 0$ such that the quadratic form $\sigma_p = \sigma_p(f, Lg, g)$ is strictly positive definite for some matrix L , where

$$\sigma_p(f, v, g) = \gamma^2(|w_1|^2 + |y - C_2 x|^2) - |\bar{C}_1 x|^2 - |u|^2 - 2 \begin{bmatrix} x \\ x_f \end{bmatrix}' p \begin{bmatrix} Ax + \bar{B}_1 w_1 + B_2 u \\ h \end{bmatrix},$$

$$f = \begin{bmatrix} w_1 \\ x \end{bmatrix}, \quad v = \begin{bmatrix} h \\ u \end{bmatrix}, \quad g = \begin{bmatrix} x_f \\ y \end{bmatrix},$$

$$L = \begin{bmatrix} A_f & B_f \\ C_f & D_f \end{bmatrix}.$$

According to the generalized Parrot's lemma, this is equivalent to positive definiteness of

$$\sigma_p(f, 0, 0) = \gamma^2(|w_1|^2 + |C_2 x|^2) - |\bar{C}_1 x|^2 - 2x' p_{11}(Ax + \bar{B}_1 w_1)$$

and

$$\sup_v \sigma_p(f, v, g)$$

for some $p = p' > 0$, where p_{11} is the upper left corner of p .

Consider first the condition of positivity of $\sigma_p(f, 0, 0)$. It is equivalent to Riccati inequality

$$\gamma^2 C_2' C_2 - \bar{C}_1' \bar{C}_1 - p_{11} A - A' p_{11} > \gamma^{-2} p_{11} \bar{B}_1 \bar{B}_1' p_{11},$$

or, equivalently,

$$\bar{B}_1 \bar{B}'_1 + AY + YA' < Y(C'_2 C_2 - \gamma^{-2} \bar{C}'_1 \bar{C}_1)Y,$$

where $Y = \gamma^2 p_{11}^{-1} > 0$. According to Theorem 7.4, existence of such Y is equivalent to existence and positive semidefiniteness of the stabilizing solution Y_∞ of (7.12). Moreover, $Y > Y_\infty$, and the difference $Y - Y_\infty$ can be made arbitrarily small whenever Y_∞ exists.

Now consider the condition of positivity of $\sup_v \sigma_p(f, v, g)$. Note that the supremum equals plus infinity unless

$$p \begin{bmatrix} x \\ x_f \end{bmatrix} = \begin{bmatrix} \psi \\ 0 \end{bmatrix},$$

in which case

$$\sigma_p(f, v, g) = \gamma^2(|w_1|^2 + |y - C_2 q_{11} \psi|^2) - |\bar{C}_1 q_{11} \psi|^2 - |u|^2 - 2\psi'(Aq_{11}\psi + \bar{B}_1 w_1 + B_2 u),$$

where q_{11} is the upper left corner of $q = p^{-1}$, and the supremum equals

$$\gamma^2(|w_1|^2 + |y - C_2 q_{11} \psi|^2) - |\bar{C}_1 q_{11} \psi|^2 - 2\psi'(Aq_{11}\psi + \bar{B}_1 w_1) + |B'_2 \psi|^2.$$

Hence, positive definiteness of the supremum is equivalent to the matrix inequality

$$q_{11} \bar{C}'_1 \bar{C}_1 q_{11} + Aq_{11} + q_{11} A' < B_2 B'_2 - \gamma^{-2} \bar{B}_1 \bar{B}'_1,$$

which can be written in an equivalent form

$$\bar{C}'_1 \bar{C}_1 + XA + A'X < X(B_2 B'_2 - \gamma^{-2} \bar{B}_1 \bar{B}'_1)X,$$

where $X = q_{11}^{-1}$. According to Theorem 7.4, existence of such X is equivalent to existence and positive semidefiniteness of the stabilizing solution X_∞ of (7.11). Moreover, $X > X_\infty$, and the difference $X - X_\infty$ can be made arbitrarily small whenever X_∞ exists.

A direct calculation of a matrix inverse shows that

$$q_{11}^{-1} = p_{11} - p_{12} p_{22}^{-1} p'_{12},$$

where

$$p = \begin{bmatrix} p_{11} & p_{12} \\ p'_{12} & p_{22} \end{bmatrix} > 0.$$

Hence

$$\gamma^2 Y^{-1} = p_{11} > q_{11}^{-1} = X \geq X_\infty.$$

Equivalently,

$$\gamma^2 I > X_\infty^{1/2} Y X_\infty^{1/2},$$

which implies (7.13). This proves necessity of the conditions for existence of a γ -suboptimal controller.

Conversely, when (7.13) is satisfied, inequality $\gamma^2 Y^{-1} > X$ will hold when the solutions X and Y of the corresponding Riccati inequalities are sufficiently close to X_∞ and Y_∞ . Then

$$p = \begin{bmatrix} \gamma^2 Y^{-1} & X - \gamma^2 Y^{-1} \\ X - \gamma^2 Y^{-1} & \gamma^2 Y^{-1} - X \end{bmatrix}$$

is positive definite and the corresponding σ_p satisfies all conditions of the generalized Parrot's lemma, which implies existence of a γ -suboptimal controller.