Massachusetts Institute of Technology

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6.245: MULTIVARIABLE CONTROL SYSTEMS

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The Waterbed Effect[1]

A common effect, usually associated with unstable zeroes and poles of the open loop plant, makes it theoretically impossible to make certain closed loop transfer functions "small" simultaneously at all frequencies: if amplitude of the frequency response is reduced in one part of the spectrum, it may have to get larger in the other part. This effect, sometimes called the *waterbed effect*, can be explained mathematically in terms of integral inequalities imposed on the closed loop transfer functions. In the basis of such results is the affine characterization of all possible closed loop responses, as well as the Cauchy integral relation for analytical functions.

5.1 Integral Identities for Analytical Functions

This subsection contains some introductory material on functions of complex variables.

5.1.1 Analytical Functions

Let Ω be an *open* subset of the complex plane $\mathbf{C}$. A function $f : \Omega \to \mathbf{C}$ is called *analytical on* Ω if the limit

$$f'(s) = \lim_{\delta \neq 0, |\delta| \to 0} \frac{f(s+\delta) - f(s)}{\delta}$$

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exists (and is continuous) for all $s \in \Omega$.

For example, functions $f(s) = \exp(s)$ and $f(s) = \sqrt{s}$ (where, for $\phi \in (-\pi, \pi)$, $\sqrt{r \exp(j\phi)} = \sqrt{r} \exp(j\phi/2)$) are analytical in the right half plane

$$\mathbf{C}_+ = \{s \in \mathbf{C} : \ \mathrm{Re}(s) > 0\},$$

while $f(s) = \mathrm{Re}(s)$ and $f(s) = |s|$ are not.

Important class of analytical functions on $\mathbf{C}_+$ is H_∞, which is defined as the set of all functions which are both analytical and bounded on $\mathbf{C}_+$. The class H_∞ includes, in particular, all stable proper transfer functions, and all Laplace transforms of integrable functions defined on $[0, \infty)$. Another important class is H_2, which consists of all analytical functions $f : \ \mathbf{C}_+ \to \mathbf{C}$ such that

$$\sup_{\sigma > 0} \int_{-\infty}^{\infty} |f(\sigma + j\omega)|^2 d\omega < \infty.$$

5.1.2 The Cauchy Identity

Assume that Ω is such that a continuous one-to-one transformation of Ω into the unit disc

$$\mathbf{D} = \{s \in \mathbf{C} : \ |s| < 1\}$$

exists. Let $f : \Omega \to \mathbf{C}$ be an analytical function, and let $\phi : \ [0, 1] \to \Omega$ be a differentiable function such that $\phi(0) = \phi(1)$. Then

$$\int_0^1 f(\phi(\tau))\dot{\phi}(\tau) d\tau = 0. \tag{5.1}$$

The integral in (5.1) can be interpreted as the *contour integral*

$$\int_C f(z) dz = 0,$$

where $C = \phi([0, 1])$ is the (closed) contour traced by $\phi(t)$ as t ranges from 0 to 1.

Integral relation (5.1) is frequently combined with the "number of encirclements" identity

$$\int_0^1 \frac{\dot{\phi}(\tau) d\tau}{\phi(\tau) - z_0} = j(\theta(1) - \theta(0)), \tag{5.2}$$

where $\phi(\tau)$ does not take value z_0 and hence can be represented in the form

$$\phi(\tau) = z_0 + r(\tau) e^{j\theta(\tau)},$$

with $r(\tau) > 0, \theta(\tau)$ being continuous functions. Note that in (5.2) the quantity $(\theta(1) - \theta(0))/2\pi$ represents the number of encirclements (anticlockwise) that the path of $\phi(\tau)$ makes around z_0 as τ increases from 0 to 1.

In particular, when the path of ϕ encircles z_0 exactly once, (and hence $z_0 \in \Omega$), combining (5.1) and (5.2) yields the well-known Cauchy identity

$$f(z_0) = \frac{1}{2\pi j} \int_C \frac{f(z)dz}{z - z_0}.$$

5.1.3 The Parceval Identity

When working with continuous time systems, the most important integral relation for analytical functions appears to be the *Parceval identity.*

Let $f \in L^2(\mathbf{R})$, i.e. $f : \mathbf{R} \to \mathbf{C}$ is a function which is square integrable over $(-\infty, \infty)$. Then the *Fourier transform* $F = F(j\omega) \in L^2(j\mathbf{R})$ of $f = f(t)$ exists in the sense that the integral

$$\int_{-\infty}^{\infty} \left| F(j\omega) - \int_{-T}^{T} e^{-j\omega t} f(t) dt \right|^2 d\omega$$

converges to zero as $T \to \infty$. If $F, G \in L^2(j\mathbf{R})$ are the Fourier transforms of $f, g \in L^2(\mathbf{R})$ then

$$\int_{-\infty}^{\infty} \bar{f}(t) g(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{F}(j\omega) G(j\omega) d\omega. \tag{5.3}$$

One can think of the Parceval identity as a special enhanced version of the Cauchy identity.

5.1.4 The class H_2

The class H_2 consists of the functions which are defined both on the right half plane, as analytical functions on C_+ which are Laplace transforms of *causal* signals $f \in L^2(\mathbf{R})$ (i.e. such that $f(t) = 0$ for $t < 0$) and on the imaginary axis $j\mathbf{R}$, as the elements of $L^2(j\mathbf{R})$ which are Fourier transforms of causal signals $f \in L^2(\mathbf{R})$.

It is important to be able to tell whether a particular function from $L^2(j\mathbf{R})$ belongs to the class H_2, without actually knowing what the inverse Fourier transform is. The following criteria is very useful: an analytical function $F : \mathbf{C}_+ \to \mathbf{C}$ is a Laplace transform of a causal signal $f \in L^2(\mathbf{R})$ if and only if there exists a finite constant c such that

$$\int_{-\infty}^{\infty} |F(\sigma + j\omega)|^2 d\omega < c \tag{5.4}$$

for all $\sigma > 0$, in which case for almost all $\omega \in \mathbf{R}$ the limit

$$F(j\omega) = \lim_{\sigma \to 0, \sigma > 0} F(\sigma + j\omega)$$

exists and equals the Fourier transform of f.

5.2 Integral Identities for Stable Transfer Functions

In this subsection, it is shown how the Cauchy formula and its analogs are applied to system analysis.

5.2.1 Special features of stable transfer functions

The condition of stability, imposed on closed loop transfer functions, is used to explain many of well-known feedback control limitations. One has to understand that, value-wise, stable transfer functions are very "redundant". (For example, behavior of such function in a neighborhood of a single point defines it in a unique way.) In feedback system design, one usually works with the following two types of statements:

- Complex frequency response of a stable transfer function can be reconstructed in the whole right half plane from the real part of its values on the imaginary axis. Similarly, the phase of a stable and *minimum phase* transfer function can be reconstructed when the system gain is known over the whole imaginary axis.
- In order for a stable transfer function $f(s)$ (or its derivative, etc.) to take large values at some points in the left half plane, the values of $|f(s)|$ on a significant portion of the imaginary axis must be large.

The statements of the first type are used to show that certain restrictions apply to the closed loop transfer functions no matter what the plant equations are. The statements of the second type are used in conjunction with the *affine interpolation constraints* characterizing closed loop transfer functions in terms of open loop unstable zeroes and poles. For the classical SISO feedback setup on Figure 5.1, where $P = P(s)$ is a given LTI CT SISO plant, and $K(s)$ is an arbitrary stabilizing controller, the closed loop sensitivity function

$$S(s) = \frac{1}{1 + P(s)K(s)}$$

must satisfy the interpolation constraints

$$S(p_i) = 0, \quad S(z_i) = 1,$$

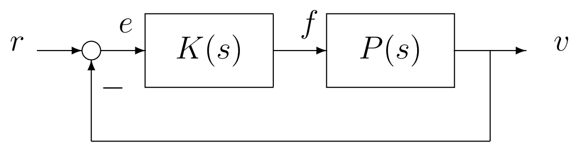

Figure 5.1: A SISO Feedback Setup

where p_i are unstable poles of P, z_i are unstable zeroes of P, and multiplicity counts.

For example, if an unstable pole p is located very close to an unstable zero z, we have $S(p) = 0$, $S(z) = 1$, and hence the derivative dS/ds must be very large somewhere near p, z. Hence, $|S(j\omega)|$ must be large on a substantial portion of the imaginary axis.

5.2.2 The Poisson Integral

For functions from the class H_2, there is a nice direct integral formula which uses the real part of its values on the imaginary axis to reconstruct the actual values in the right half plane.

Theorem 5.1 *If $F \in H_2$ then*

$$F(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\text{Re } F(j\omega)d\omega}{s - j\omega}. \tag{5.5}$$

for all $s \in \mathbf{C}_+$.

Proof Let F be the Fourier transform of f. Note that f is causal and define $g(t) = f(t) + f(-t)$. Then, by the definition of $F(s)$,

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt = \int_{-\infty}^{\infty} g(t)\bar{r}(t)dt,$$

where

$$r(t) = u(t)e^{-\bar{s}t},$$

and $u(\cdot)$ denotes the unit step function. Hence, by the Parceval formula,

$$F(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(j\omega)\bar{R}(j\omega)d\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\text{Re } F(j\omega)d\omega}{s - j\omega},$$

since the Fourier transform $R(j\omega)$ of $r(t)$ is $R(j\omega) = 1/(j\omega + \bar{s})$ and the Fourier transform $G(j\omega)$ of g is $G(j\omega) = 2\text{Re } F(j\omega)$. ■

When looking separately for the real and imaginary part of $F(s)$, (5.5) yields

$$\text{Re } F(a+bj) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{a\text{Re } F(j\omega)d\omega}{a^2+(\omega-b)^2}, \tag{5.6}$$

$$\text{Im } F(a+bj) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(\omega-b)\text{Re } F(j\omega)d\omega}{a^2+(\omega-b)^2}. \tag{5.7}$$

Important versions of the Poisson formula are obtained when F is defined as a *logarithm* $F(s) = \log H(s)$ of a minimum-phase stable transfer function.

5.2.3 Bode Gain and Phase Relation

We say that a stable rational transfer function $H(s)$ is *minimum phase* if $H(s) \neq 0$ for $\text{Re}(s) > 0$, and $H(1) > 0$. For non-rational transfer functions the definition is not so straightforward. In general a function $H(s)$ from class H_∞ is called “minimum phase” if $H(1) > 0$ and any function $F(s)$ from class H_2 can be approximated arbitrarily well in the mean square sense by the products $H(s)Q_k(s)$, where $Q_k(s)$ are stable rational transfer functions. It can be shown that $H(s) \neq 0$ in the right half plane for any minimum phase function $H(s) \in H_\infty$. However, there are functions from H_∞ which have no unstable zeros, but are still not minimum phase, such as $H(s) = \exp(-s)$. On the other hand, a function can have a zero on the imaginary axis, but still be minimum-phase such as $H(s) = s/(s+1)$.

The following statement is known as the *Bode's Gain/Phase relation*.

Theorem 5.2 *If $L \in H_\infty$ is a minimum-phase transfer function, then the phase of L is uniquely defined by its gain, according to the formula:*

$$\text{phase}(L(j\omega)) = \int_{-\infty}^{\infty} \frac{d\log|L(e^\nu\omega)|}{d\nu}\psi(\nu)d\nu,$$

where

$$\psi(\nu) = \frac{1}{\pi}\log\frac{e^{|\nu|/2}+e^{-|\nu|/2}}{e^{|\nu|/2}-e^{-|\nu|/2}}.$$

Since $\psi(\nu) \geq 0$ and $\psi(\nu) \ll 1$ for $\nu \gg 1$, the main contribution to the integral is made in the region $\nu \approx 0$. Hence the integral mainly depends on the values of

$$\frac{d\log|L(e^\nu\omega)|}{d\nu}$$

with $\nu \approx 0$, i.e. essentially on $(d/d\omega)\log L(j\omega)$.

5.2.4 Bode's Sensitivity Integral

The following relation is an example of an inequality that does not allow the closed loop sensitivity to be small on the imaginary axis.
Theorem (Bode's Sensitivity Integral) LET S BE A STABLE RATIONAL TRANSFER FUNCTION SUCH THAT $1 - S$ HAS RELATIVE DEGREE GREATER THAN 1. THEN

$$\int_0^\infty \log |S(j\omega)| d\omega = \pi \sum_{k=1}^{m} \mathrm{Re}(z_k)$$

WHERE z_k ARE THE UNSTABLE ZEROS OF S.

Here the relative degree condition ensures that $S(s) \approx 1$ for large $|s|$. On the other hand, $S(z_k) = 0$. This means large variation of S in the right half plane, and an inequality bounding $\log |S(j\omega)|$ from below in an "integral" sense results.

5.3 A Case Study

In this section, it is demonstrated how the Poisson formula (5.6) can be used to show that some design specifications are practically infeasible.

5.3.1 Formulation of the Problem

Consider the standard SISO feedback design setup shown on Figure 5.1, where $P(s)$ is the given open loop plant model, and $K(s)$ is the stabilizing controller to be designed. We will assume that $P(s)$ has an unstable zero at $s = 2$ and an unstable pole at $s = 3$. According to the classical control, the unstable zero will limit the closed loop bandwidth, in the sense that thre closed-loop sensitivity transfer function $S = S(s)$ (from r to e) cannot have small gain on the frequency interval $\omega \in [0, \omega_0]$ when $\omega_0 \gg 2$. Contrary to this, the mathematical theory tells us that, unless P has zeros on the imaginary axis, for every $\epsilon > 0$ and for every $\omega_0 > 0$ there exists a stabilizing controller $C(s)$ such that $|S(j\omega)| < \epsilon$ for all $\omega \in [0, \omega_0]$.

To reconcile these two statements, one can expect that every controller which *theoretically* provides a very large closed loop bandwidth achieves this at the expense of producing very bad behavior at other frequencies. To show that this is indeed the case, let us bound from below the H-Infinity norm of S assuming that $|S(j\omega)|$ does not exceed 0.1 for $|\omega| < 10$ rad/sec.

5.3.2 Using the Poisson Formula

From the problem formulation we know that S is a stable transfer function, $S(2) = 1$, and $S(3) = 0$. Let

$$B(s) = \frac{s+p_1}{s-p_1} \cdot \frac{s+p_2}{s-p_2} \cdot \dots \cdot \frac{s+p_n}{s-p_n},$$

where p_k are the strictly unstable zeros of $S(s)(s+3)/(s-3)$ (multiplicity counts). Let

$$S_m(s) = S(s)B(s)\frac{s+3}{s-3}.$$

Note that

(a) $S_m(s)$ does not have strictly unstable zeros or poles;

(b) $|S_m(j\omega)| = |S(j\omega)|$ for all $\omega \in \mathbf{R}$;

(c) $|S_m(2)| \geq 5$,

where (a) follows by construction, and (b),(c) are the consequence of the transfer functions $B_p(s) = (s-p)/(s+p)$, with $\mathrm{Re}(p) > 0$, being all-pass, which means $|B_p(j\omega)| = 1$ for $\omega \in \mathbf{R}$ and $|B_p(s)| \leq 1$ for $\mathrm{Re}(s) > 0$.

By (a), the Poisson integral formula can be applied to $\log\, S_m$, which yields

$$\log |S_m(a+jb)| = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{a \log |S_m(j\omega)| d\omega}{(\omega - b)^2 + a^2}$$

for all $a > 0$, $b \in \mathbf{R}$. Setting $a = 2$, $b = 0$, and using (b),(c), we get

$$\log(5) \leq \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{2 \log |S(j\omega)| d\omega}{\omega^2 + 4}.$$

Equivalently, the change of the independent variable $\omega := 2\omega$ yields

$$\log(5) \leq \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log |S(2j\omega)| d\omega}{\omega^2 + 1} = \frac{2}{\pi}\int_{0}^{\infty} \frac{\log |S(2j\omega)| d\omega}{\omega^2 + 1}.$$

Remember that, by the problem formulation,

$$|S(2j\omega)| \leq 0.1 \ \text{ for } \ \omega < 5.$$

Also let

$$M = \sup_{\omega > 5} |S(2j\omega)|.$$

Taking into accound that

$$\int_{\omega_1}^{\omega_2} \frac{d\omega}{1+\omega^2} = \arctan(\omega_2) - \arctan(\omega_1),$$

we have

$$\begin{aligned}
\frac{\pi \log(5)}{2} &\leq \int_0^\infty \frac{\log|S(2j\omega)|d\omega}{\omega^2+1} \\
&= \int_0^5 \frac{\log|S(2j\omega)|d\omega}{\omega^2+1} + \int_5^\infty \frac{\log|S(2j\omega)|d\omega}{\omega^2+1} \\
&\leq \log(0.1)\arctan(5) + \log(M)(\frac{\pi}{2} - \arctan(5)).
\end{aligned}$$

Hence

$$\log(M) \geq \frac{\pi \log(5)/2 + \arctan(5)\log(10)}{\frac{\pi}{2} - \arctan(5)} \approx 250dB.$$

The calculation was done using MATLAB instruction

```
20*(pi*log10(5)/2+atan(5))/(pi/2-atan(5))
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Indeed, this is a very large *lower bound* of the H-Infinity norm, which makes the design objective $|S(j\omega)| \leq 0.1$ for $\omega \leq 10$ practically infeasible.