Massachusetts Institute of Technology

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6.245: MULTIVARIABLE CONTROL SYSTEMS

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Using H2 Optimization[1]

This lecture provides some useful hints on ways to apply H2 optimization.

4.1 Closed Loop Properties of H2 Optimal Systems

Recall that for the standard setup of minimizing the closed loop H2 norm from w to z in

$$\dot{x} = Ax + B_1 w + B_2 u, \tag{4.1}$$
$$z = C_1 x + D_{12} u, \tag{4.2}$$
$$y = C_2 x + D_{21} w, \tag{4.3}$$

the optimal LTI controller is given by

$$u = K\hat{x}, \tag{4.4}$$
$$\dot{\hat{x}} = A\hat{x} + B_2 u + L(C_2\hat{x} - y), \tag{4.5}$$

where K is such that

$$|C_1 p + D_{12} q|^2 + 2p' P_{fi}(Ap + B_2 q) = (q - Kp)' R_z (q - Kp), \tag{4.6}$$

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$A + B_2K$ is a Hurwitz matrix, $R_z = D'_{12}D_{12}$, and $P_{fi} = P'_{fi} \geq 0$ is a symmetric matrix. Similarly, L is such that

$$|B'_1p + D'_{21}q|^2 + 2p'P_{se}(A'p + C'_2q) = (q - L'p)'R_y(q - L'p), \tag{4.7}$$

where $A + LC_2$ is a Hurwitz matrix, $R_y = D_{21}D'_{21}$, and $P_{se} = P'_{se} \geq 0$ is a symmetric matrix.

In this section, we investigate the closed loop properties of H2 optimal feedback systems which can be derived from these characterizations.

4.1.1 Closed Loop Poles

The closed loop system dynamics can be written in terms of the new states $e = x - \hat{x}$ and $\hat{x}$:

$$\dot{e} = (A + LC_2)e + (B_1 + LD_{21})w, \tag{4.8}$$

$$\dot{\hat{x}} = (A + B_2K)\hat{x} - LC_2e - LD_{21}w. \tag{4.9}$$

In particular, this implies that the closed loop poles are eigenvalues of $A + B_2K$ and $A + LC_2$.

It turns out that identities (4.6), (4.7) can be used to get some insight into locations of these poles. Indeed, eigenvalues of $A + B_2K$ are zeros of

$$\psi_u(s) = I - K(sI - A)^{-1}B_2,$$

because

$$\det(sI - A - B_2K) = \det(sI - A)\det(I - B_2K(sI - A)^{-1}) = \det(sI - A)^{-1}\det(\psi_u(s)),$$

where we have use the fact that

$$\det(I + ab) = \det(I + ba)$$

for every pair of matrices a, b of compatible dimensions (and the identity matrices on both sides may have different dimensions). Similarly, eigenvalues of $A + LC_2$ are zeros of

$$\psi_y(s) = I - L'(sI - A')^{-1}C'_2.$$

On the other hand, noting that (4.6) will hold for complex vectors p, q, as long as it is modified to a Hermitian norm identity

$$|C_1p + D_{12}q|^2 + 2\mathrm{Re}\{p'P_{fi}(Ap + B_2q)\} = (q - Kp)'R_z(q - Kp), \tag{4.10}$$

where $'$ denotes Hermitian conjugation, and substituting

$$p = (j\omega I - A)^{-1} B_2 q$$

yields the identity

$$P_{12}(-s)^T P_{12}(s) = \psi_u(-s)^T R_z \psi_u(s) \tag{4.11}$$

for all $s = j\omega$ on the imaginary axis (except, strictly speaking, possible eigenvalues of A). Since(4.11) is an identity between rational functions, its validity on the imaginary axis implies its validity for all s. In particular, *zeros of* $\det(\psi_u(s))$ *(i.e. the eigenvalues of* $A + B_2K$*) are the* stable *zeros of* $\det\{P_{12}(-s)^T P_{12}(s)\}$.

A similar derivation shows that *zeros of* $\det(\psi_y(s))$ *(i.e. the eigenvalues of* $A + LC_2$*) are the* stable *zeros of* $\det\{P_{21}(s)P_{21}(-s)^T\}$.

For example, in a special case when the optimization setup is defined by a SISO plant

$$P_0(s) = C(sI - A)^{-1} B,$$

where (A, B) is controllable and (C, A) is observable, according to

$$\dot{x} = Ax + B(w_1 + u), \tag{4.12}$$

$$z_1 = Cx, \tag{4.13}$$

$$z_2 = r_z u, \tag{4.14}$$

$$y = Cx + r_y w_2, \tag{4.15}$$

where $w = [w_1; w_2]$ is the total disturbance, and r_z, r_y are positive coefficients, the closed loop poles of H2 optimal feedback system are the stable zeros of

$$\phi_u(s) = r_z^2 + P_0(s)P_0(-s)$$

and

$$\phi_y(s) = r_y^2 + P_0(s)P_0(-s).$$

In particular, one can now apply the standard properties of the root locus to predict the asymptotic behavior of the poles as r_z and r_y converge to zero or infinity. For example, when r_z is small, some eigenvalues of $A + B_2K$ are close to the stable zeros of $P_0(s)$ and to the mirror (with respect to the imaginary axis) images of unstable zeros of $P_0(s)$, with the rest of eigenvalues approaching infinity. Similarly, when r_z is large, the eigenvalues of $A+BK_2$ approximate stable eigenvalues of A and the mirror images of unstable eigenvalues of A.

4.1.2 Closed Loop Stability Robustness

Is stability of the closed loop be preserved when the optimal gains K, L are replaced by some other gains? Optimal H2 controllers frequently have plenty of robustness this way, though this usually does not transform into real robustness (which is caused by errors in modeling the *outside* world, not the controller's coefficients).

Consider again the special feedback optimization setup (4.12)-(4.15). The corresponding solutions P_{fi} and P_{se} of Riccati equations satisfy

$$P_{fi}(A+BK)+(A+BK)'P_{fi}=-C'C-r_z^2K'K,\ B'P_{fi}=-r_z^2K, \tag{4.16}$$

$$P_{se}(A'+L'C')+(A'+L'C')P_{se}=-BB'-r_y^2LL',\ C_2P_{se}=-r_y^2L', \tag{4.17}$$

and hence both are positive definite. Moreover (4.16) implies that

$$P_{fi}(A+BzK)+(A+BzK)'P_{fi}=-C'C-r_z^2K'(1+2\mathrm{Re}(z))K$$

is negative definite for every complex number z with $\mathrm{Re}(z)>-0.5$. Therefore, all eigenvalues of $A+BzK$ have negative real part for $\mathrm{Re}(z)>-0.5$. Similarly, $A+LzC_2$ is a Hurwitz matrix for $\mathrm{Re}(z)>-0.5$.

This observation was originally used to claim that the optimal H2 feedback has a 50 percent gain margin and a 60 percent phase margin. In reality, this only holds for the full information or state estimation problems. Indeed, K enters the formula for the output feedback controller in two places: in defining the control output $u=K\hat{x}$, and in defining the state observer dynamics

$$\dot{\hat{x}}=(A+B_2K+LC_2)\hat{x}-Ly.$$

While an *unknown* controller gain uncertainty can be combined with K in the first appearance, in most situations one cannot do the same in the second appearance. The situation with the observer gain is similar. All the controversy around the originally claimed "robustness" of optimal H2 control has served as a catalyst for developing H-Infinity optimization and mu-synthesis.