

Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science
6.245: MULTIVARIABLE CONTROL SYSTEMS

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Interpretations for Standard Optimization Setup¹

This is the second lecture on standard feedback optimization setup. It describes a variety of ways to come up with H2 and H-Infinity performance measures, provides hints for reducing non-standard objectives to the standard format.

2.1 Systems and signals background

This section provides some minimal background in systems and signals needed for understanding this lecture.

2.1.1 Signals and systems in continuous time

It is convenient to think of continuous time (CT) *signals* as real vector-valued functions of time $t \in [0, \infty)$, integrable over any bounded interval. From this viewpoint, $f_1(t) = t^{-1/2}$ (defined, for the sake of mathematical accuracy, as zero at $t = 0$) and $f_2(t) = e^{t^2}$ are signals, while $f_3(t) = t^{-1}$ (where $f_3(0) = 0$) and $f_4(t) = \delta(t)$ (Dirac delta) are not. The set of all signals with values in \mathbf{R}^k will be denoted by \mathcal{L}^k .

A continuous time *system* S with a k -dimensional input and m -dimensional output is simply a map $S : \mathcal{L}^k \mapsto \mathcal{L}^m$ (usually multi-valued, so that one input $f \in \mathcal{L}^k$ corresponds

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to many possible outputs $g \in \mathcal{L}^m$). For example, the familiar pure integrator system (transfer function $1/s$) maps a signal $f \in \mathcal{L}^1$ to signals of the form

$$g(t) = c_0 + \int_0^t f(\tau) d\tau,$$

where c_0 is an arbitrary constant. In general, a system's output is not necessarily unique because it may also depend on a set of auxiliary parameters (e.g. the initial *states* of the system), as in Figure 2.1. In the case of the pure integrator system described above,

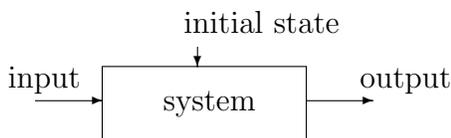


Figure 2.1: System with initial conditions

constant c_0 plays the role of an initial state.

2.1.2 Signals and systems in discrete time

Let us view discrete time (DT) signals as continuous time signals which only change value at a discrete set of uniformly spaced time instances $t_k = kT$, where $T > 0$ is a fixed real number called the *sampling rate* of a DT signal, and k is a non-negative integer. The usual meaning of a discrete time signal is that at every time $t \geq 0$ it represents the *last available sample* of a continuous time signal, provided the samples are taken uniformly with interval T starting at zero time. For example, sampling CT signal $f(t) = \cos(\pi t)$ at rate $T = 1$ yields a discrete-time signal

$$f_d(t) = \begin{cases} 1, & k \leq t < k+1, k \in \{0, 2, 4, \dots\}, \\ -1, & k \leq t < k+1, k \in \{1, 3, 5, \dots\} \end{cases}.$$

Note that this f_d can also be viewed as a DT signal at sampling rate $1/M$ for every positive integer M (though, indeed, it will not be the result of sampling $f(t) = \cos(\pi t)$ at rate $T = 1/M$ for $M \neq 1$).

An alternative way of representing a discrete time signal $f = f(t)$ with sampling rate $T > 0$ is by specifying T and the sequence of its values $f[k]$ at time instances $t_k = kT$, i.e.

$$f[k] = f(kT), \quad k = 0, 1, 2, \dots$$

Thus, a DT signal $f(t)$ is completely defined by its sampling rate T and by the sequence of sampled values $f[k] = f(kT)$. For example, the DT signal $f_d(t)$ from above can be defined as such with sampling rate $T = 1$ and sampled values sequence $f_d[k] = (-1)^k$.

The set of all discrete time k -dimensional signals at sampling rate T will be denoted by $\mathcal{L}_{[T]}^k$.

A discrete time system S is a map $S : \mathcal{L}_{[T]}^k \mapsto \mathcal{L}_{[T]}^m$ (usually multi-valued). Note that this definition requires same sampling rates for input and output. Therefore, a DT system with a k -dimensional input and m -dimensional output can also be viewed as a map $S : l_+^k \mapsto l_+^m$, where l_+^q denotes the set of all sequences $f = f[k]$ of q -dimensional real vectors, indexed by non-negative integers k . For example, the familiar one step delay system (transfer function $1/z$) maps $f = f[k]$ into $g[k] = f[k - 1]$, with the value of $g[0]$ being arbitrary.

2.1.3 Finite Order CT LTI Models

A state-space model for a finite order CT LTI system H with input $f = f(t)$, output $g = g(t)$, and state $x = x(t)$ has the form

$$\dot{x}(t) = Ax(t) + Bf(t), \quad (2.1)$$

$$g(t) = Cx(t) + Df(t), \quad (2.2)$$

where A, B, C, D are constant matrices with real entries.

$$H = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)$$

frequently serves as a shortcut notation. Given an input $f = f(t)$, the output $g = g(t)$ is defined by the initial state vector $x(0)$, according to the formula

$$g(t) = Ce^{At}x(0) + Df(t) + \int_0^t Ce^{A\tau}Bf(t - \tau)d\tau.$$

The *transfer matrix* (transfer *function* in the case when both f and g are scalar) of the system is defined for all complex s such that $sI - A$ is invertible by

$$H(s) = D + C(sI - A)^{-1}B.$$

2.1.4 Finite Order DT LTI Models

A state-space model for a finite order DT LTI system H with input $f = f(t)$, output $g = g(t)$, and state $x = x(t)$ has the form

$$x(t+T) = Ax(t) + Bf(t), \quad (2.3)$$

$$g(t) = Cx(t) + Df(t), \quad (2.4)$$

where A, B, C, D are constant matrices with real entries. Here f, g, x are DT signals with same sampling rate $T > 0$. In terms of sampled value sequences $f[k] = f(kT)$, $x[k] = x(kT)$, and $g[k] = g(kT)$, equations have the form

$$x[k+1] = Ax[k] + Bf[k], \quad (2.5)$$

$$g[k] = Cx[k] + Df[k], \quad (2.6)$$

The *transfer matrix* (transfer *function* in the case when both f and g are scalar) of the system is defined for all complex z such that $zI - A$ is invertible by

$$H(z) = D + C(zI - A)^{-1}B.$$

2.2 Interpretations for H-Infinity and H2 norms

H-Infinity and H2 norms are frequently used as the cost measure in feedback optimization. This section describes interpretations of the norms as performance measures.

2.2.1 H-Infinity and H2 norms for finite order LTI systems

Let

$$H = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)$$

be a finite order state space CT LTI model. The model is called *stable* if A is a Hurwitz matrix, i.e. if all eigenvalues of A have negative real part.

The *H-Infinity norm* $\|H\|_\infty$ of H is defined as the supremum (minimal upper bound) of the largest singular number of its transfer function over the imaginary axis:

$$\|H\|_\infty = \sup_{\omega \in \mathbf{R}} \sigma_{\max}(H(j\omega)),$$

where

$$H(s) = D + C(sI - A)^{-1}B,$$

and, for an k -by- m complex matrix M ,

$$\sigma_{\max}(M) = \max_{u \in \mathbf{C}^m, |u|=1} |Mu|,$$

and $|v|$ denotes the standard Hermitean norm (length) of vector v .

If H is a discrete time system, maximization over the imaginary axis is replaced with maximization over the unit circle:

$$\|H\|_{\infty} = \sup_{\omega \in [-\pi, \pi]} \sigma_{\max}(H(e^{j\omega})).$$

The H_2 norm $\|H\|_2$ of a finite order stable CT LTI system H with $D = 0$ is defined by the integral

$$\|H\|_2^2 = \int_0^{\infty} \text{trace}(h(t)h(t)') dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace}(H(j\omega)H(j\omega)') d\omega,$$

where

$$h(t) = Ce^{At}B$$

is the impulse response matrix of H .

In the discrete time case (where $D \neq 0$ is allowed), the H_2 norm is defined by

$$\|H\|_2^2 = \text{trace}(D'D) + \sum_{k=0}^{\infty} \text{trace} B'(A')^k C' C A^k B = \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{trace}(H(e^{j\omega})H(e^{j\omega})') d\omega.$$

2.2.2 H-Infinity norm as L2 gain

The L2 (strictly speaking, “L2-to-L2”) gain of a continuous time system S with input f and output g is defined as the minimal $\gamma \geq 0$ such that

$$\inf_{T \geq 0} \int_0^T \{\gamma^2 |f(t)|^2 - |g(t)|^2\} dt > -\infty$$

for all input/output pairs $g = S(f)$ where input f is square integrable over arbitrary finite intervals. The definition for discrete time systems is similar, with integrals replaced by sums:

$$\inf_{N \geq 0} \sum_{k=0}^N \{\gamma^2 |f[k]|^2 - |g[k]|^2\} > -\infty,$$

where $g = S(f)$.

The informal rationale behind the definition is as follows: for “zero initial conditions” (whatever this means), we expect the “energy” of the output to be bounded by the energy of the input times the L2 gain squared. Since non-zero initial conditions can produce non-zero output even for zero input, the actual definition says that the difference between the energies must be bounded on one side.

L2 gain is a key concept in robustness analysis. The importance of the H-Infinity norm is largely due to the fact that, for a stable finite order LTI system, H-Infinity norm equals L2 gain.

Theorem 2.1 *L2 gain of a stable finite order LTI system equals its H-Infinity norm.*

Proof Consider the continuous time case (the DT case is similar). Let $H = H(s)$ be the transfer function of the system.

To show that L2 gain cannot be larger than H-Infinity norm, use the Parseval theorem. Consider the case of zero initial conditions first. For an arbitrary input signal f and for $T > 0$ let f_T denote the signal defined by

$$f_T(t) = \begin{cases} f(t), & t < T, \\ 0, & t \geq T. \end{cases}$$

Let g and g^T denote the response of the system to f and f_T respectively, both assuming zero initial conditions. Then f_T, g^T are square integrable over $t \in (0, \infty)$ (for g^T this is true since A is a Hurwitz matrix), and hence both have Fourier transforms \tilde{f}_T and \tilde{g}^T respectively. In addition, by causality, $g(t) = g^T(t)$ for $t < T$. Hence

$$\begin{aligned} \int_0^T |g(t)|^2 dt &= \int_0^T |g^T(t)|^2 dt \leq \int_0^\infty |g^T(t)|^2 dt \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty |\tilde{g}^T(j\omega)|^2 d\omega = \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty |H(j\omega)\tilde{f}_T(j\omega)|^2 d\omega \\ &\leq \frac{1}{2\pi} \int_{-\infty}^\infty \|H\|_\infty^2 |f_T(j\omega)|^2 d\omega = \|H\|_\infty^2 \int_0^\infty |f_T(t)|^2 dt = \|H\|_\infty^2 \int_0^T |f(t)|^2 dt. \end{aligned}$$

In the case of non-zero initial conditions the total system response is given by $g = g_0 + g_1$, where g_0 is the zero state response and g_1 is zero input response. Since A is a Hurwitz matrix, g_0 is square integrable over $t \in (0, \infty)$. Hence, for every $\epsilon > 0$,

$$\int_0^T |g(t)|^2 dt \leq (1 + \epsilon) \int_0^T |g_0(t)|^2 dt + (1 + \frac{1}{\epsilon}) \int_0^T |g_1(t)|^2 dt$$

$$\leq (1 + \epsilon) \|H\|_\infty^2 \int_0^T |f(t)|^2 dt + (1 + \frac{1}{\epsilon}) \int_0^\infty |g_1(t)|^2 dt.$$

To show that H-Infinity norm cannot be larger than L2 gain, consider zero initial conditions and sinusoidal inputs $f(t) = f_0 \cos(\omega t)$, where frequency ω and vector f_0 are appropriately chosen. ■

2.2.3 H2 norm and L2-to-L-Infinity gain

L2-to-L-Infinity gain of a stable state space model is defined as the supremum of the amplitude of its time domain response to an input signal of unit energy.

Theorem 2.2 *L2-to-L-Infinity gain of a stable LTI system with a scalar output equals its H2 norm.*

Proof Consider the continuous-time case (the DT case is similar).

To show that H2 norm is not smaller than the L2-to-L-Infinity gain, use the standard Cauchy-Schwartz inequality:

$$|y(T)|^2 = \left| \int_0^T h(t) f(T-t) dt \right|^2 \leq \int_0^T |h(t)|^2 dt \int_0^T |f(t)|^2 dt.$$

Since the inequality becomes equality for $f(t) = h(T-t)'$, the L2-to-L-Infinity gain actually equals the H2 norm. ■

2.2.4 H2 norm and variance of white noise response

H2 norm of a system is also an measure of system sensitivity to white noise input.

Remember that the “weak sense” white noise $f = f[k]$ in discrete time is a sequence of random variables $f[k]$ with zero mean, unit variance, and zero cross-corellation:

$$\mathbf{E}f[k] = 0, \quad \mathbf{E}f[k_1]f[k_2]' = \delta(k_1, k_2)I,$$

where $\delta(k_1, k_2) = 0$ for $k_1 \neq k_2$ and $\delta(k, k) = 1$ for all k .

The continuous time white noise is a slightly more complicated concept: it is a *generalized* random process $f = f(t)$ (something akin the Dirac delta in the world of deterministic functions), which can be characterized by its effect in integration: if

$$\xi = \int_{t_1}^{t_2} h(t) f(t) dt,$$

where $h = h(t)$ is a row vector of appropriate length, then

$$\mathbf{E}\xi = 0, \quad \mathbf{E}|\xi|^2 = \int_{t_1}^{t_2} |h(t)|^2 dt.$$

Combining this information with the definition of H2 norm, one can conclude that, for a stable LTI system, *the asymptotic (as $t \rightarrow \infty$) variance of white noise response equals square of H2 norm.*

2.3 Small gain theorems

Small gain theorems (the well-known one for H-Infinity norms, and the less known one for H2 norm) are among the main reasons to minimize there performanc measures.

2.3.1 A small gain theorem for H-Infinity

While, formally speaking, H-Infinity optimization designs LTI controllers for LTI systems, it can be used to design controllers for nonlinear or time-varying systems. The main idea is to extract all non-LTI components of a model, and put them into an external block, which will affect the LTI part through an input, and will in turn be driven by an output of an LTI system, as shown on Figure 2.2, left, where Δ is the nonlinear/time varying block, and G is the LTI system.

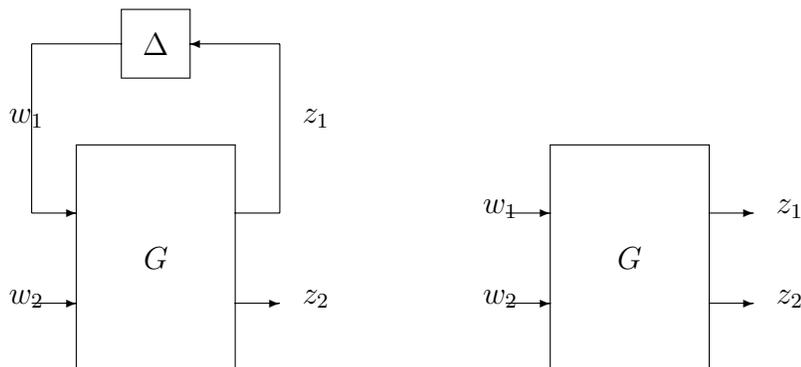


Figure 2.2: Small gain theorem

The following theorem, a version of the famous *small gain condition*, allows one to estimate the L2 gain of the nonlinear feedback system on the left of Figure 2.2 based on the L2 gain properties of Δ and G .

Theorem 2.3 *If G and Δ separately have L2 gains less than one then the feedback interconnection on Figure 2.2, left, has L2 gain (from w_2 to z_2) less than 1.*

Proof Consider the signals consistent with the left diagram of Figure 2.2. By assumption about Δ ,

$$\inf_{T>0} \int_0^T \{|z_1|^2 - |w_1|^2\} dt > -\infty.$$

By assumption about G , for some $r \in (0, 1)$,

$$\inf_{T>0} \int_0^T \{r|w_1|^2 + r|w_2|^2 - |z_1|^2 - |z_2|^2\} dt > -\infty.$$

Combining these two conditions yields

$$\inf_{T>0} \int_0^T \{r|w_2|^2 - |z_2|^2\} dt > -\infty.$$

■

2.3.2 Example: L2 gain optimization via small gain theorem

Consider the task of designing a feedback controller for the standard setup shown on Figure 2.3, where G is an unstable LTI plant with delay,

$$P(s) = \frac{e^{-\tau s}}{s-1},$$

and F is the controller to be designed to guarantee good tracking of the reference input r at lower frequencies.

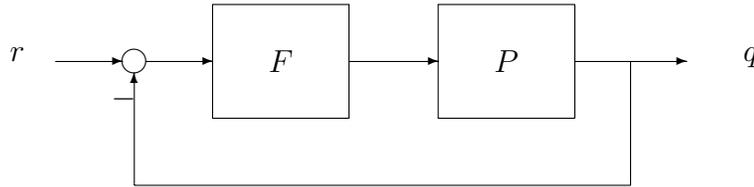


Figure 2.3: Feedback design with delay

We begin by approximating the stable part of P by a lower order transfer function, while bounding the approximation error:

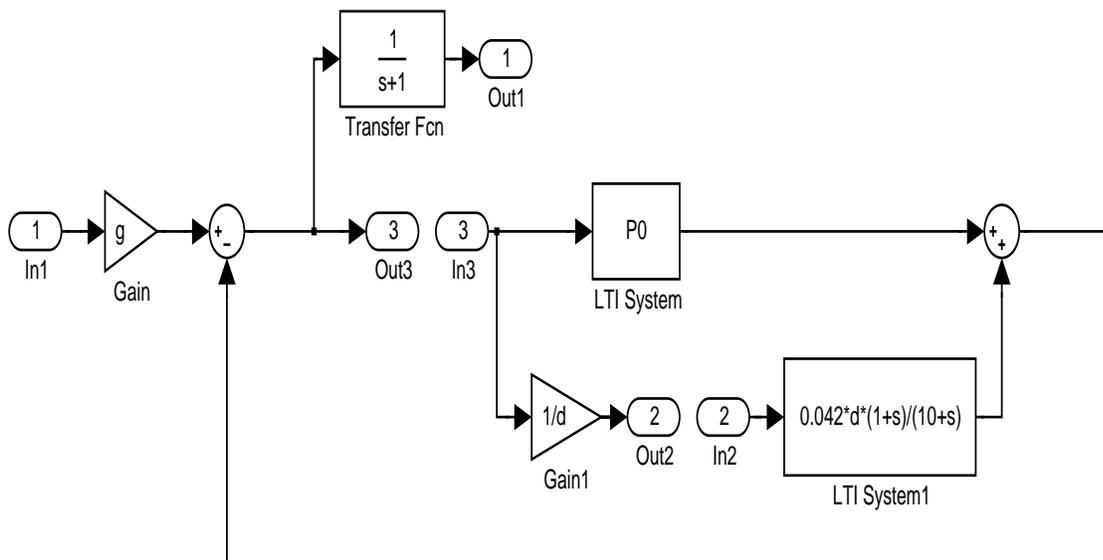
$$P(s) = P_0(s) + W(s)\Delta(s),$$

where

$$P_0(s) = \frac{e^{-\tau}}{s-1} - \frac{\tau}{1+0.5\tau} \frac{1}{1+0.5\tau s},$$

$$W(s) = 0.042 \frac{1+s}{10+s},$$

and $\Delta(s)$ is known to have H-Infinity norm not larger than one. The corresponding feedback design diagram are shown below.



```
tau=0.1;
s=tf('s');
G0=-(tau/(1+tau/2))/(1+tau*s/2);
w=linspace(0,100,10000);
G0w=squeeze(freqresp(G0,w)).';
Gw=(exp(-tau*j*w)-exp(-tau))./(j*w-1);
max(abs((G0w-Gw).*(1+j*w/10)./(1+j*w)))
```

```
P0=exp(-tau)/(s-1)+G0;
g=5;
d=150;
[A,B,C,D]=linmod2('lec2_ex1a');
p=pck(A,B,C,D);
k=hinfsyn(p,1,1,0,20,0.01);
```

2.3.3 A small gain theorem for H2

H2 norm, too, can serve as a measure of robustness to white noise perturbations of the feedback loop gain.

Remember that the usual small gain bound on a CT system Δ with input f and output g is an inequality of the form

$$\int_0^T |g(t)|^2 dt \leq \text{const} + \gamma^2 \int_0^T |f(t)|^2 dt,$$

where the constant may depend on the input/output pair (f, g) . A similar definition can be given in the discrete time case. When dealing with randomized signals and systems, the L2 gain bounding condition can be replaced by its “expected value” version

$$\int_0^T \mathbf{E}|g(t)|^2 dt \leq \text{const} + \gamma^2 \int_0^T \mathbf{E}|f(t)|^2 dt,$$

and it is easy to show that the small gain theorem (leading to an H-Infinity norm bound imposed on the nominal LTI system as a condition of overall stability and performance) still holds.

Consider now a very special case of a gain-bounded randomized “uncertain” block Δ . Let us call a random CT column vector signal g *uncorrelated* if it has zero mean and

$$\mathbf{E} \left| \int_{t_1}^{t_2} p(t)g(t)dt \right|^2 = \int_{t_1}^{t_2} |p(t)|^2 \mathbf{E}|g(t)|^2 dt$$

for every square integrable row vector function $p = p(t)$. In the discrete time case, simply require that

$$\mathbf{E} \left| \sum_{k=k_1}^{k_2} p[k]g[k] \right|^2 = \sum_{k=k_1}^{k_2} |p[k]|^2 \mathbf{E}|g[k]|^2$$

for every sequence $p = p[k]$ of row vectors.

Theorem 2.4 *Let G be a stable finite order LTI system with two inputs w_1, w_2 and two outputs z_1, z_2 (w_i, z_i can be vectors). Let G_{ij} denote the system describing dependence of z_i on w_j . Assume that w is uncorrelated, $E|w_1(t)|^2 = 1$ for all t , $\|G_{22}\|_2^2 < 1$, and*

$$\int_0^T \mathbf{E}|w_2(t)|^2 dt \leq \text{const} + \int_0^T \mathbf{E}|z_2(t)|^2 dt,$$

for all T . Then

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T \mathbf{E}|z_1(t)|^2 dt \leq \|G_{11}\|_2^2 + \frac{\|G_{12}\|_2^2 \cdot \|G_{21}\|_2^2}{1 - \|G_{22}\|_2^2}.$$

2.3.4 Example: analysis via H2 small gain

Consider a simple example which can be easily solved analytically with or without using the small gain argument. Consider the randomized first order discrete time dynamical system

$$y[k+1] = (a + bv_2[k])y[k] + v_1[k],$$

where a, b are known real coefficients, $v = [v_1; v_2]$ is a *strict sense white noise* signal, i.e.

$$\mathbf{E}v_i[k]^2 = 1, \quad \mathbf{E}v_i[k] = 0,$$

and it is known that $v_i[k]$ is independent of $y[t], v_i[t-1]$ for all $t \leq k$. What is the asymptotic value of $\mathbf{E}y[k]^2$ as $k \rightarrow \infty$?

This question can be answered easily without relying on the H2 small gain theorem, because

$$\mathbf{E}y[k+1]^2 = (a^2 + b^2)\mathbf{E}y[k]^2 + 1,$$

which immediately implies that the asymptotic value of $\mathbf{E}y[k]^2$ is infinity unless $a^2 + b^2 < 1$, in which case

$$\limsup_{k \rightarrow \infty} \mathbf{E}y[k]^2 = \frac{1}{1 - a^2 - b^2}.$$

To apply the H2 small gain theorem in this case, use

$$z_1[k] = z_2[k] = y[k], \quad w_1[k] = v_1[k], \quad w_2[k] = v_2[k]y[k].$$

Then z and w are related through an LTI transformation with transfer matrix

$$G = G(z) = \begin{bmatrix} \frac{1}{z-a} & \frac{b}{z-a} \\ \frac{1}{z-a} & \frac{b}{z-a} \end{bmatrix}.$$

Assuming $|a| < 1$,

$$\|G_{11}\|_2^2 = \|G_{21}\|_2^2 = \frac{1}{1 - a^2}, \quad \|G_{12}\|_2^2 = \|G_{22}\|_2^2 = \frac{b^2}{1 - a^2}.$$

By the assumptions made about $v_i[k]$, signal $w = [w_1; w_2]$ is uncorrelated, and $\mathbf{E}w_2[k]^2 = \mathbf{E}z_2[k]^2$ for all k . Hence the small H2 gain theorem applies, and yields the upper bound

$$\limsup_{k \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \mathbf{E}z_1[k]^2 \leq \|G_{11}\|_2^2 + \frac{\|G_{12}\|_2^2 \cdot \|G_{21}\|_2^2}{1 - \|G_{22}\|_2^2} = \frac{1}{1 - a^2 - b^2}.$$