Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.245: MULTIVARIABLE CONTROL SYSTEMS

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Analysis of Uncertain Systems[1]

In this lecture enhanced techniques of stability and robustness analysis of LTI systems are developed by combining the ideas of *small gain theorem*, *zero exclusion principle*, and *relaxations in quadratic programming*

13.1 Motivation and Basic Definitions

Basic principles of LTI system robustness analysis are presented in this subsection.

13.1.1 Small Gain Theorem/Circle Criterion

The circle criterion can be applied to systems which are non-linear, time-varying, infinite-dimensional, etc. Consider the feedback interconnection from Figure 13.1, where G and Δ are arbitrary causal systems with finite L2 norms $\|G\|$ and $\|\Delta\|$ respectively, such that the feedback interconnection is well-posed (i.e. for any input signals f_1, f_2 the feedback equations have a solution z_1, z_2 which depends causally on z_1, z_2). The following theorem was proven in previous lectures.

Theorem 13.1 *If* $\|\Delta\| \cdot \|G\| < 1$*, the feedback interconnection is stable.*

The small gain condition is a simple but general tool for certifying stability and robustness of uncertain systems, linear and nonlinear. However, it becomes excessively

[1]Version of April 21, 2004

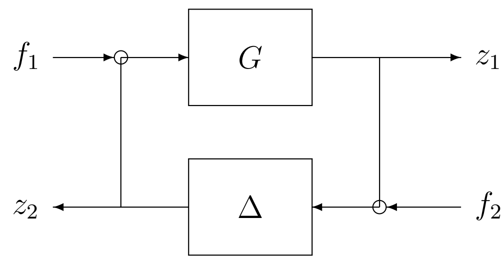

Figure 13.1: setup for the Small Gain Theorem

conservative when some *structural* information about Δ is available. Consider, for example, the case when $G(s) = s/(s^2+s+1)$ and $\Delta = \Delta_0 < 0$ is a constant negative feedback gain. Then $\|G\| = \|G\|_\infty = 1$, but the feedback system remains stable for arbitrarily large values of Δ. Here the additional condition imposed on the *structure* of Δ (in this case, on its *phase*) yields a much better robustness margin than predicted by the small gain theorem.

13.1.2 Zero Exclusion Principle

A finite order LTI system is stable if it does not have poles in the closed right half plane

$$\mathbf{C}_+ = \{s \in \mathbf{C} : \ \mathrm{Re}(s) \geq 0\}.$$

However, checking the *whole* $\mathbf{C}_+$ for the absense of poles is usually inconvenient, especially when the system (and its poles as well) is uncertain. Therefore it is very important to establish that analysis of stability of uncertain LTI systems can be based on excluding marginal instability.

Theorem 13.2 *Let $G = G(s)$, $\Delta = \Delta(s)$ be LTI systems with finite L2 gains. Assume that*

$$|\det(I - \tau G(j\omega)\Delta(j\omega))| \geq \epsilon > 0 \quad \text{for all} \quad \omega \in \mathbf{R}, \ \tau \in [0,1].$$

Then the feedback interconnection of G and Δ has finite L2 gain.

Theorem 13.2 implies the following version of the *zero exclusion principle*. Let G be a stable finite order LTI system with m inputs and n outputs. Let $\mathbf{\Delta}$ be a *cone* of complex m-by-n complex matrices, i.e. a closed set such that

$$\bar{\Delta} \in \mathbf{\Delta}, \ \lambda \geq 0 \ \text{ implies } \ \lambda\bar{\Delta} \in \mathbf{\Delta}.$$

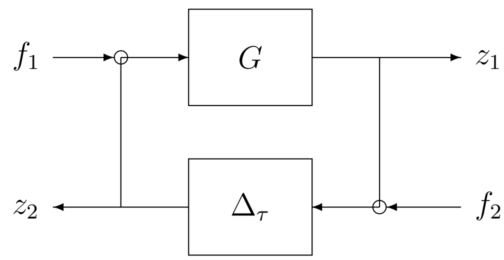

Figure 13.2: a homotopy argument in stability analysis

The interconnection of G and Δ is stable for any transfer function Δ satisfying the condition $\Delta(j\omega) \in \mathbf{\Delta}$ if and only if

$$\det(I - G(j\omega)\bar{\Delta}) \neq 0 \quad \forall\, \omega \in \mathbf{R} \cup \{\infty\}, \ \bar{\Delta} \in \mathbf{\Delta}.$$

13.1.3 Structured Singular Values

Let $\mathbf{\Delta}$ be a cone of complex n-by-m matrices. The *structured singular value*

$$\mu = \mu(M) = \mu(M, \mathbf{\Delta}) = \mu_{\mathbf{\Delta}}(M)$$

is defined for any complex m-by-n matrix as the reciprocal of the smallest norm of $\Delta \in \mathbf{\Delta}$ such that $I - M\Delta$ is *not invertible*. Here $\mathbf{\Delta}$ defines the *structure* of an uncertain block Δ such that $\Delta(j\omega) \in \mathbf{D}$: the smaller $\mathbf{\Delta}$ is, the more structure, and the smaller $\mu(M, \mathbf{\Delta})$ is going to be.

Note that, when $\mathbf{\Delta}$ is the set of *all* n-by-m matrices (i.e. when there is "no structure"), we have

$$\mu(M, \mathbf{\Delta}) = \|M\| = \sigma_{\max}(M)$$

which explains the term "structured singular value"

The so-called "complex structured singular value" $\mu_{\mathbf{C}}(M)$ corresponds to the case when

$$\mathbf{\Delta} = \left\{ \begin{bmatrix} \Delta_1 & 0 & & 0 \\ 0 & \Delta_2 & & \\ & & \ddots & \\ 0 & & & \Delta_n \end{bmatrix} : \ \Delta_i \in \mathbf{C} \right\}$$

is the set of all complex diagonal matrices.

The so-called "real structured singular value" $\mu_{\mathbf{R}}(M)$ corresponds to the case when

$$\boldsymbol{\Delta} = \left\{ \begin{bmatrix} \delta_1 & 0 & & 0 \\ 0 & \delta_2 & & \\ & & \ddots & \\ 0 & & & \delta_n \end{bmatrix} : \ \delta_i \in \mathbf{R} \right\}$$

is the set of all real diagonal matrices.

M. Safonov uses notation $K_m(M, \boldsymbol{\Delta}) = 1/\mu(M, \boldsymbol{\Delta})$.

13.1.4 Examples

For

$$M = \begin{bmatrix} j & 100 \\ 0 & j \end{bmatrix}$$

we have

$$\sigma_{max}(M) \approx 100, \quad \mu_{\mathbf{C}}(M) = 1, \quad \mu_{\mathbf{R}}(M) = 0$$

If $\boldsymbol{\Delta} = \{\Delta = \delta I : \ \ \delta \in \mathbf{C}\}$, then $\mu(M, \boldsymbol{\Delta})$ is the spectral radius of M (i.e. the largest absolute value of an eigenvalue of M).

$$\mu_{\mathbf{R}}\left(\begin{bmatrix} 1 & t \\ -t & 1 \end{bmatrix}\right) = \begin{cases} 0, & t \neq 0, \\ 1, & t = 0. \end{cases}$$

In particular, this demonstrates that some versions of μ are discontinuous functions of its argument. However, $\mu_{\mathbf{C}}(M)$ can be proven to be a continuous function of M.

13.1.5 A "Small μ Theorem"

Consider a feedback interconnection of a given LTI system $G(s)$ and an uncertain LTI system $\Delta(s)$, where $\Delta(s)$ is any *stable* LTI system such that $\Delta(j\omega) \in \boldsymbol{\Delta}$ for all ω, and $\|\Delta(s)\|_\infty < 1$. Then, by the definition of $\mu(G, \boldsymbol{\Delta})$, and by the "zero exclusion principle", the interconnection is stable for any expected $\Delta(s)$ if

$$\sup_{\omega} \mu_{\boldsymbol{\Delta}}(\bar{G}) < 1$$

for any $\bar{G}$ which can be approximated arbitrarily well by the values of $G(j\omega)$.

For example, in the case of "structured" unmodeled dynamics, shown on Figure 13.3, the interconnection is stable and has worst case w_0 to z_0 gain less than one if and only if

$$\sup_{\omega} \mu_{\mathbf{C}}(G(j\omega)) < 1.$$

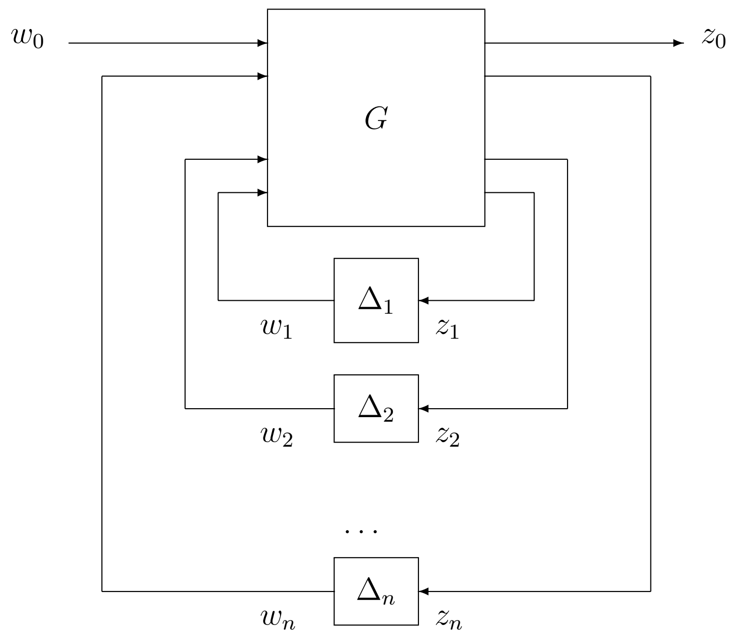

Figure 13.3: small complex mu theorem

13.2 Computation of robustness margins

In general, μ is difficult to compute exactly (such problems are called "NP hard" by computer scientists). In practice, computable upper and lower bounds of μ are used. When those bounds are far apart, a "branch and bound" technique is used.

An *upper bound* of μ is a function $\hat{\mu}$ such that $\hat{\mu}(M) \geq \mu(M)$ for any M. Upper bounds of μ give *sufficient* conditions of stability and robust performance. Lower bounds of μ prove that certain systems are not robustly stable. Usually, calculation of a large lower bound of μ comes with an example of a destabilizing uncertainty.

13.2.1 Lower Bounds of μ

Standard lower bounds for μ are obtained by finding a local minimum in the non-convex optimization problem

$$\rho(\Delta M) \to \max_{\Delta \in \mathbf{\Delta}}$$

where $\rho(A)$ is the spectral radius of A (when $\mathbf{\Delta}$ is invariant with respect to multiplication by complex scalars) or the real spectral radius of A (for the "real" versions of μ). Roughly speaking, the search for low bounds of μ amounds to simulations of the uncertain systems with different values of the uncertainty parameters.

For most common structures $\mathbf{\Delta}$, the search over $\mathbf{\Delta}$ can be replaced by the search over the *unitary* elements in $\mathbf{\Delta}$, i.e. such that $\Delta\Delta' = I$.

For example, for a 3-by-3 matrix M, figuring out whether $\mu_{\mathbf{R}}(M) < 1$ is equivalent to checking that $I - M\Delta$ is invertible for any diagonal matrix with elements $\delta_i \in [-1, 1]$ on the diagonal ($i = 1, 2, 3$).

Consider separately the 2 cases: $\delta_1 \geq 0$ and $\delta_1 \leq 0$. Since

$$[0, 1] = 0.5 + 0.5 \cdot [-1, 1],$$

$$[-1, 0] = -0.5 + 0.5 \cdot [-1, 1],$$

in each case, the problem can be reduced to checking that $\mu_{\mathbf{R}}(M_\pm) < 1$, where

$$M_\pm = (I \pm 0.5Me_1e_1')^{-1}M(I - 0.5e_1e_1')$$

e_1 is the first basis vector, "+" corresponds to $\delta_1 \in [0, 1]$, "−" corresponds to $\delta_1 \in [-1, 0]$. For each of $M_\pm$, we expect that the gap between upper and lower bounds of μ will be smaller, since the actual range of δ_i has been reduced.

13.2.2 Quadratic Constraints

Matrix $I - M\Delta$ is not invertibe if and only if the system of equations

$$y = Mw, \; w = \Delta y$$

has a non-zero solution (y, w). To show that this is impossible for $\Delta \in \mathbf{\Delta}$, $r\|\Delta\| \leq 1$, we start with finding quadratic forms $\sigma = \sigma(y, w)$ such that

$$\sigma(y, \Delta y) \geq 0 \;\; \forall \; \Delta \in \mathbf{\Delta}, \; \|\Delta\| \leq 1$$

Such conditions are called *quadratic constraints* describing the relation $w = \Delta y$. The idea of "describing" uncertainty using quadratic inequalities is the background of most robustness criteria.

Let Σ be a set of quadratic forms $\sigma(y, w)$ such that

$$\sigma(y, \Delta y) \geq 0 \;\; \forall \; \Delta \in \mathbf{\Delta}, \; \|\Delta\| \leq 1$$

$$\sigma(0, w) < 0 \;\; \text{for } w \neq 0$$

Note that any convex combination of such quadratic forms satisfies the condition as well.

Define a functional on Σ by

$$J(\sigma) = \inf\{r \geq 0 : \; \sigma(Mw, rw) < 0 \;\; \forall \; w \neq 0$$

- Functional J is quasi-convex.
- The infimum of J over Σ is an upper bound of $\mu(M, \mathbf{\Delta})$.
- The larger Σ, the better the upper bound.

This is the idea behind upper bounds of μ.

13.2.3 Elementary Uncertainty

Here we derive quadratic constraints for elementary components of uncertainty structures

Unmodeled Dynamics If $w = \Delta y$, where Δ is an arbitrary complex matrix with $\|\Delta\| \leq 1$, the relation between w and y is described by

$$\sigma(y, w) = \|y\|^2 - \|w\|^2 \geq 0$$

Repeated Real Scalar If $w = \delta y$, where $\delta \in [-1, 1]$ is a real number, the relation between w and y is described by

$$\sigma(y, w) = y'Dy - w'Dw + 2\mathrm{Re}(y'Sw) \geq 0$$

where where D, S are arbitrary matrices such that $D = D' \geq 0$ and $S = -S'$.

Quadratic constraints for other elementary uncertainty relations are obtained in a similar way.

13.2.4 An Upper Bound for μ

Quadratic constraints for a general structured uncertainty are obtained as convex combinations of the "elementary" constraints.

Let $\mathbf{\Delta} = \{\Delta\}$ be the uncertainty structure for which $w = \Delta y$ means

$$y = [y_1^r; \dots; y_N^r; y_1^c; \dots; y_M^c]$$

$$w = [w_1^r; \dots; w_N^r; w_1^c; \dots; w_M^c]$$

where

$$w_k^r = \delta_k y_k^r, \;\; y_k^r, w_k^r \in \mathbf{C}^{n(k)}, \;\; \delta_k \in \mathbf{R}$$

$$w_k^c = \Delta_k y_k^c, \;\; y_k^c \in \mathbf{C}^{m(k)}, \;\; w_k^c \in \mathbf{C}^{p(k)}$$

An upper bound of $\mu(M, \mathbf{\Delta})$ is given by the minimal $r > 0$ such that

$$M'DM + MS + S'M < r^2 \hat{D}$$

where

$$S = \begin{bmatrix} \hat{S} & 0 \\ 0 & 0 \end{bmatrix},$$

$$D = \mathrm{diag}\{D_1, \ldots, D_N, d_1 I_{m(k)}, \ldots, d_M I_{m(M)}\}$$

$$\hat{D} = \mathrm{diag}\{D_1, \ldots, D_N, d_1 I_{p(k)}, \ldots, d_M I_{p(M)}\}$$

$$\hat{S} = \mathrm{diag}\{S_1, \ldots, S_M\}$$

matrices $D_k = D_k' \geq 0$ and $S_k = -S_k'$ have size $n(k)$-by-$n(k)$. Optimization of $r_{\min}$ over $D, \hat{D}, S$ is quasi-convex.

The upper bound above was obtained using the quadratic constraint

$$\sigma(y, \Delta y) \geq 0 \quad \forall\ \Delta \in \mathbf{\Delta},\ \|\Delta\| \leq 1$$

where

$$\begin{aligned} \sigma(y, w) \quad = \quad & \sum_{k=1}^{N} y_k^{r\prime} D_k y_k^r - w_k^{r\prime} D_k w_k^r \\ & +2\mathrm{Re}(y_k^{r\prime} S_k w_k^r) \\ & + \sum_{k=1}^{M} d_i (\|y_k^c\|^2 - \|w_k^c\|^2) \end{aligned}$$

13.3 Basic Definitions of IQC Analysis

Integral Quadratic Constraints are used to express some information about subsystems of a larger system, in a way that is convenient to use in the analysis of the whole. We will start with defining mathematical models of signals, systems, system states, subsystems, and IQC.

13.3.1 Signals

A *signal* is a real vector-valued function $f:\ (0, \infty) \mapsto \mathbf{R}^m$ of time t, $0 < t < \infty$, which is square integrable on every *finite* interval $\{t\} = (0, T)$, $0 < T < \infty$. Thus, for our purposes, functions such as $(t-1)^{1/3}$ or $f(t) = \exp(t^2)$ are valid models of signals, while $f(t) = \delta(t)$ or $f(t) = (t-1)^{-1}$ are not. The set of all m-vector signals will be referred to as $L_{2,loc}^m$. The subset of $L_{2,loc}^m$ consisting of all signals which are square integrable over the infinite time interval $(0, \infty)$ will be denoted by L_2^m.

By default, a signal is treated as a force-like quantity, i.e. its effect is always defined in terms of its integral. Therefore, two signals f_1, f_2 are considered equal if $f_1(t) = f_2(t)$ for *almost all* $t \in (0, \infty)$, i.e. for all $t \in (0, \infty)$ except, possibly, a set of zero Lebesque measure. In particular, looking at the value $f(t_0)$ of a given signal at a given single moment of time $t_0 \in (0, \infty)$ does not make much sense, unless $f(\cdot)$ has some special properties. For example, if it is known that $f, g \in L^m_{2,loc}$ are related by the condition

$$f(t_2) - f(t_1) = \int_{t_1}^{t_2} g(t)dt$$

for almost all $t_1, t_2 \in (0, \infty)$ (which is a mathematically clean way of saying that $df/dt = g$), then the value of the *essential limit*

$$\operatorname{ess} \lim_{t \to t_0} f(t)$$

can be used as a meaningful interpretation of $f(t_0)$.

While many sources would define signals as functions defined for all $t \in (-\infty, \infty)$, it is better to avoid this because some system models cannot be traced back in time to $t = -\infty$. For example, the non-zero solutions of the autonomous differential equation $dx/dt = -x^3$ cannot be extended backwards in time till $t = -\infty$. Another such example is the model of Brownian motion.

When $f \in L^m_{2,loc}$ is a signal, referring to $f(t)$ makes formal sense only for $t > 0$. For convenience, it will be assumed that $f(t) = 0$ for all $t \leq 0$, $f \in L^m_{2,loc}$.

13.3.2 Systems

At some very general level, the word *system* means just a relation between signals. Following this observation, a *system* will be defined as a *set of signals* of a fixed dimension, i.e. as a subset of $L^m_{2,loc}$ for some m.

A common way of defining systems is by differential equations. For example, the set of differential equations

$$\dot{x}(t) = F(x(t), v(t)), \quad w(t) = H(x(t), v(t)),$$

where $F: \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$ and $H: \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^k$ are two given functions can be used to refer to an *input/output system model* $\mathcal{S}_{io}$ defined as the set of all $z = [v; w] \in L^{m+k}_{2,loc}$ for which there exists $x \in L^n_{2,loc}$ such that $w(t) = H(x(t), v(t))$, $g(t) = F(x(t), w(t))$ is locally integrable over every interval (t_1, t_2) with $0 < t_1 < t_2 < \infty$), and $dx/dt = g$. Alternatively, the equations can refer to a *state-space system model* $\mathcal{S}_{ss}$ defined as the set of all $z = [v; w; x] \in L^{m+k+n}_{2,loc}$ for which $w(t) = H(x(t), v(t))$, $g(t) = F(x(t), w(t))$ is locally integrable over every interval (t_1, t_2) with $0 < t_1 < t_2 < \infty$), and $dx/dt = g$.

13.3.3 System State

By its meaning, the current *system state* determines the set of all possible extensions of system signals into past or future. For a general system $\mathcal{S}$, it is more reasonable to introduce the notion of two signals $z_1, z_2 \in \mathcal{S}$ defining the *same state* of $\mathcal{S}$ at a given time instance $T > 0$.

Let $\mathcal{S} \subset L_{2,loc}^m$ be a system. Two signals $z_1, z_2 \in \mathcal{S}$ are said to *commute in* $\mathcal{S}$ *at time* $T \in (0, \infty)$ if $h_1, h_2 \in \mathcal{S}$, where

$$h_1(t) = \begin{cases} z_1(t), & t < T, \\ z_2(t), & t > T, \end{cases} \quad h_2(t) = \begin{cases} z_2(t), & t < T, \\ z_1(t), & t > T. \end{cases}$$

In other words, two signals commute if their pasts and futures can be interchanged without violating the underlying system constraints. Two signals $z_1, z_2 \in \mathcal{S}$ are said to *define same state of* $\mathcal{S}$ *at time* $T \in (0, \infty)$ if the set of all signals $g \in \mathcal{S}$ which commute with z_1 at time T equals the set of all signals $g \in \mathcal{S}$ which commute with z_2 at time T.

Example 13.1 Let

$$\mathcal{S} = \{z = [f; g] \in L_{2,loc}^2 : \ df/dt = g\}$$

be the model of a pure integrator. Note that f must be continuous for every $z = [f; g] \in \mathcal{S}$. Therefore, if two signals $z_1 = [f_1; g_1] \in \mathcal{S}$ and $z_2 = [f_2; g_2] \in \mathcal{S}$ commute in $\mathcal{S}$ at time T then $f_1(T) = f_2(T)$ (otherwise the composite of f_1 and f_2 will be discontinuous at time T). On the other hand, by inspection, the composites h_1 and h_2 of z_1 and z_2 will belong to $\mathcal{S}$ as long as $f_1(T) = f_2(T)$. Therefore $z_1 = [f_1; g_1] \in \mathcal{S}$ and $z_2 = [f_2; g_2] \in \mathcal{S}$ define same state of $\mathcal{S}$ at time T if and only if $f_1(T) = f_2(T)$.

In other words, it would be correct to declare $f(T)$ to be the state of $z = [f; g] \in \mathcal{S}$ in $\mathcal{S}$ at time T. However, declaring $f(T)^3 - \exp(T)$ as the state at time T would be an equally valid choice.

13.3.4 Subsystems

Typically, a *subsystem* of a given system is defined by a subset of constraints describing the whole behavior. In terms of sets, this means that the signal set defining a system must be a *subset* of the signal set defining its subsystem. It turns out, however, that the notion of a subsystem must satisfy some additional properties to be useful in system analysis.

Let $\mathcal{S} \subset \mathcal{S}_0 \subset L_{2,loc}^m$. $\mathcal{S}_0$ is called a *subsystem* of $\mathcal{S}$ if every two signals $z_1, z_2 \in \mathcal{S}$ which define same state of $\mathcal{S}$ at time $T > 0$ also define same state of $\mathcal{S}_0$ at that time.

For example, system

$$\mathcal{S}_0 = \{z = [y; f] \in L_{2,loc}^2 : \ dy/dt = df/dt\}$$

is a subsystem of

$$\mathcal{S}_1 = \{z = [y; f] \in L^2_{2,loc} : \; dy/dt = df/dt, f = 0\},$$

but is *not* a subsystem of

$$\mathcal{S}_2 = \{z = [y; f] \in L^2_{2,loc} : \; dy/dt = df/dt, f = y\}.$$

One interpretation of this example is that the feedback interconnection of systems $dy/dt = df/dt$ and $f = 0$ is well-posed, but the feedback interconnection of systems $dy/dt = df/dt$ and $f = y$ is *not* well-posed.

13.3.5 Integral Quadratic Constraints

Let σ be a quadratic form on $\mathbf{R}^m$, i.e.

$$\sigma(z) = z'\Sigma z \quad \forall \; z \in \mathbf{R}^m,$$

where $\Sigma = \Sigma'$ is a symmetric m-by-m matrix. System $\mathcal{S} \subset L^m_{2,loc}$ is said to *satisfy the IQC defined by* σ if there exists a function $V : \; \mathcal{S} \times (0, \infty) \to \mathbf{R}$ such that

(a) $V(z_1, T) = V(z_2, T)$ whenever z_1 and z_2 define same state of $\mathcal{S}$ at time $T > 0$;

(b) the inequality
$$\liminf_{T \to \infty} V(z, T) \geq 0 \tag{13.1}$$
holds for all $z \in \mathcal{S}$ which are square integrable over $(0, \infty)$;

(c) the inequality
$$\int_{t_1}^{t_2} \sigma(z(t))dt \geq V(z, t_2) - V(z, t_1) \tag{13.2}$$
holds for all $t_2 > t_1 > 0$,

in which case one can write $\sigma \geq \dot{V}$ as a shortcut.

In the definition of IQC, (a) means that $V(z, T)$ depends only on the state defined by z at time T; (b) means (informally) that V is non-negative when the system state "approaches zero"; (c) is the actual IQC inequality declaring that $\sigma(z(T))$ is an upper bound of the time derivative of $V(z, t)$ at $t = T$.

Example 13.2 Let $\phi : \mathbf{R} \mapsto \mathbf{R}$ be a monotonic function such that $\phi(0) = 0$. For a fixed $a > 0$ define

$$\psi(y) = \frac{1}{a} \int_0^{ay} \phi(\tau) d\tau.$$

Consider system

$$\mathcal{S} = \{z = [v; w; x] \in L^3_{2,loc} : \ w = \phi(v), \ dx/dt = -ax + v\}.$$

Let

$$V(z, T) = \psi(x(T)) \ \text{ where } \ z = [v; w; x] \in \mathcal{S}.$$

It is easy to check that V satisfies conditions (a),(b) in the definition, and (c) holds for $\sigma([v; w; x]) = w(v - ax)$. Therefore, $w(v - ax) \geq \dot{V}$ is a valid IQC describing $\mathcal{S}$.

Further analysis of the situation suggests that the IQC actually describes the *memoryless* relation between v and $w = \phi(v)$. The additional variable x is defined by a first-order LTI transformation of v. In general, it is usually quite beneficial to consider such LTI "extensions" of a given system model, since it allows one to produce more of useful IQC for a given relation. A shortcut for the IQC derived above is

$$w' \left(\frac{s}{s + a} v \right) \geq \dot{V},$$

where $\left(\frac{s}{s+a} v\right)$ denotes x, emphasizing that the IQC describes a relation between v and $w = \phi(v)$, while x serves as an auxiliary variable.

13.4 Basic Theorems of IQC Analysis

Typically, IQC analysis of a system $\mathcal{S} = \{z\} \subset L^m_{2,loc}$ comes through three different phases.

In the first phase, *IQC modeling*, a set of IQC

$$\sigma_k(z) \geq \dot{V}_k, \quad k = 1, 2, \dots$$

is derived for $\mathcal{S}$. Since an IQC for a subsystem holds automatically for the whole system, it helps to recognize subsystems of $\mathcal{S}$ for which some IQC have already been established. For example, extensive sets of IQC are available for LTI models, memoryless models, uncertain models, etc. Note that, in order to get a rich set of IQC, it is usually necessary to re-define z (and hence $\mathcal{S}$) by introducing auxiliary variables defined as LTI transformations of the original components of $z \in \mathcal{S}$. To complete an IQC model, two linear functions of z, $f = L_{in} z$ and $y = L_{out} z$, are to be designated respectively as the external noise and the transient response to it (for autonomous systems, the natural choice would be $f = 0$).

In the second phase, *IQC feasibility analysis*, one tries to find a convex combination

$$\sigma(z) = \sum_k c_k \sigma_k(z), \quad c_k \geq 0, \quad \sum_k c_k > 0$$

of the quadratic forms σ_k which satisfies the inequality

$$\sigma(z) \leq \gamma |f|^2 - |y|^2 = \gamma |L_{in} z|^2 - |L_{out} z|^2 \quad \forall\, z \in \mathbf{R}^m,$$

where γ is as small as possible. If such convex combination does not exist, the attempt at IQC analysis is declared a failure (which possibly indicates that a larger set of IQC should have been used). Otherwise, the IQC model is declared feasible, and analysis comes into the third phase, *IQC post-feasibility analysis* when one has to determine whether the corresponding convex combination

$$V = \sum_k c_k V_k,$$

is non-negative. Note that $V \geq 0$ would imply the *L2 gain* inequality

$$\int_T^\infty |y(t)|^2 dt \leq V(z, T) + \gamma \int_T^\infty |f(t)|^2 dt,$$

which can be interpreted as input/output stability of $\mathcal{S}$, while having $V(z, T) < 0$ for some $z \in \mathcal{S}$ such that $f(t) = 0$ for $t > T$ implies

$$\lim_{T \to \infty} \inf_{\tau > T} V(z, \tau) < 0,$$

which can be interpreted as the system state not converging to zero despite the disturbance being identically zero after time T.

The general mathematical statements presented in this section are used in the three phases of IQC analysis.

13.4.1 IQC for LTI Relations

Almost any system model includes some LTI relations between signals. Therefore, it is important to know which IQC are implied by LTI relations. The following theorem answers this question.

Let A and B be real matrices of dimensions n-by-n and n-by-k. Let

$$\mathcal{S}_{A,B} = \{z = [x; u] \in L_{2,loc}^{n+k} : \ \frac{dx}{dt} = Ax + Bu$$

be the LTI system defined by A and B. Let $\Sigma = \Sigma'$ be a real symmetric $(n+k)$-by-$(n+k)$ matrix, and $\sigma(x, u) = \sigma(z) = z'\Sigma z$ for $z = [x; u]$.

Theorem 13.3 *Assume that the pair (A, B) is stabilizable. Then IQC $\sigma(z) \geq \dot{V}$ holds on $\mathcal{S}_{A,B}$ if and only if there exists a symmetric real n-by-n matrix $P = P'$ such that*

$$\sigma(x, u) \geq 2x'P(Ax + Bu) \quad \forall\ x \in \mathbf{R}^n,\ u \in \mathbf{R}^k. \tag{13.3}$$

Since the expression $2x'P(Ax + Bu)$ equals the derivative of $V_P(x) = x'Px$ subject to the system equation $dx/dt = Ax + Bu$, the theorem has the following interpretation: an IQC $\sigma \geq \dot{V}$ holds on $\mathcal{S}_{A,B}$ for *some* storage function V if and only if the IQC $\sigma \geq \dot{V}_P$ (with the same σ) holds for some *quadratic* storage function V_P.

Proof The sufficiency of (13.3) follows from the unsurprising observation that two signals $z_1 = [x_1; u_1]$ and $z_2 = [x_2; u_2]$ define same state of $\mathcal{S}_{A,B}$ at time $T > 0$ if and only if $x_1(T) = x_2(T)$. Hence $V_P(z, T) = x(T)'Px(T)$ is a function of the state of $z = [x; u]$ at time T. In addition, if both $z = [x; u] \in \mathcal{S}_{A,B}$ is square integrable over $(0, \infty)$ then $x(T) \to 0$ as $T \to \infty$. Hence $\sigma \geq \dot{V}_P$ is a valid IQC for $\mathcal{S}_{A,B}$.

To prove the necessity of (13.3), note that an IQC $\sigma \geq \dot{V}$ implies that

$$\int_T^\infty \sigma(z)dt \geq -V(z, T) = -V(x(T), T)$$

whenever $z = [x; u] \in \mathcal{S}_{A,B}$ is square integrable over $(0, \infty)$. Hence the infimum

$$U(a) = \inf\{\int_T^\infty \sigma(x, u)dt : \ [x; u] \in \mathcal{S}_{A,B} \cap L_2^{n+k},\ x(T) = a\}$$

(which, by the time invariance of the system equations, does not depend on $T > 0$) is finite for all $a \in \mathbf{R}^n$. According to the Bellman principle of dynamic programming, the function $\tilde{V}(a) = -U(a)$ satisfies

$$\int_{t_1}^{t_2} \sigma(x,u)dt \geq \tilde{V}(x(t_2)) - \tilde{V}(x(t_1))$$

for all $z = [x; u] \in \mathcal{S}_{A,B}$. On the other hand, $U = U(a)$, as an infimum of a quadratic form on a vector space subject to the linear constraint $x(T) = a$ must be a quadratic form with respect to the parameter $a \in \mathbf{R}^n$. Hence $\tilde{V}(a) = a'Pa$ for some $P = P'$, and (13.3) follows. ■

13.4.2 The KYP Lemma

The Kalman-Popov-Yakubovich (KYP) Lemma answers the following question: given A, B, and Σ, defined as in the previous subsection, when does there exist a symmetric real $P = P'$ such that (13.3) holds? More precisely, the lemma reduces the question to a frequency-domain equivalent, thus establishing a link between state-space and frequency domain viewpoints.

Note that the the definition $\sigma(z) = z'\Sigma z$ also applies to complex vectors $z \in \mathbf{C}^{n+k}$, where $'$ stands for Hermitian conjugation.

Theorem 13.4 *Assume that the pair (A, B) is stabilizable, there exists at least one $s \in \mathbf{C}$ such that $sI - A$ is invertible, and the matrix $\Sigma_s = \Sigma_s'$ of the Hermitian form*

$$\tilde{\sigma}_s(u) = \sigma((sI - A)^{-1}Bu, u) = u'\Sigma_s u$$

is non-singular as well. Then (13.3) holds for some real matrix $P = P'$ if and only if

$$\sigma(x,u) \geq 0 \quad \text{for all} \quad x \in \mathbf{C}^n, u \in \mathbf{C}^k, \omega \in \mathbf{R}: \; j\omega x = Ax + Bu. \qquad (13.4)$$

Possibly the most important theorem of linear system theory, the KYP Lemma (called "positive real lemma" in some versions) has many different proofs, all of which are too long to include in this text. However, the necessity of (13.4) follows immediately from the observation that $\sigma_P(x,u) = 0$ for all x, u such that $j\omega x = Ax + Bu$. One proof of sufficiency follows the path of the proof of Theorem 13.3: first, the Parceval formula is used to show boundedness of some infimum in a linear-quadratic optimization problem, and then P emerges as the matrix of the quadratic form defining the dependence of the infimum on the initial conditions.

In this lecture, the KYP lemma will be used to give a convenient frequency domain interpretation of IQC feasibility analysis for a simplified system analysis setup described in the next subsection.

13.4.3 IQC Analysis in the Frequency Domain

Consider the typical system analysis setup shown on Figure 13.4. Here G is a known

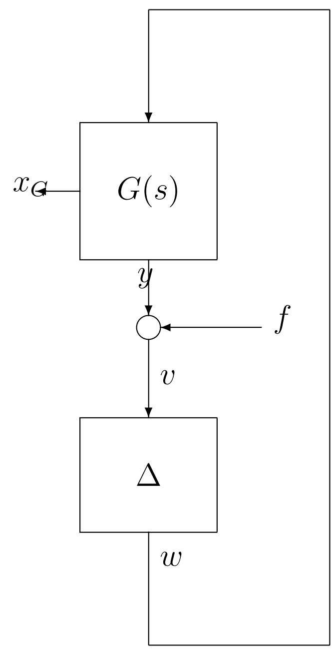

Figure 13.4:

stable finite order LTI model with input w, state x_G, and output y, described by the equations

$$\dot{x}_G(t) = A_G x_G(t) + B_G w(t), \quad y(t) = C_G x_G(t) + D_G w(t) \tag{13.5}$$

with known real coefficient matrices A_G, B_G, C_G, D_G. Signal f model external disturbance, and Δ represents a relation between v and w which completes the constraints imposed on system dynamics.

Assume that an IQC

$$\sigma_\Delta(x_L, v_L, w_L) \geq \dot{V}_\Delta \tag{13.6}$$

holds for the relation between v and w, where x_G is the auxiliary signal introduced as the state of a stable LTI system L with inputs v and w, as shown on Figure 13.5. The equations of L are given by

$$\dot{x}_L(T) = A_L x_L(t) + B_L^v v(t) + B_L^w w(t), \quad x_L(0) = 0, \tag{13.7}$$

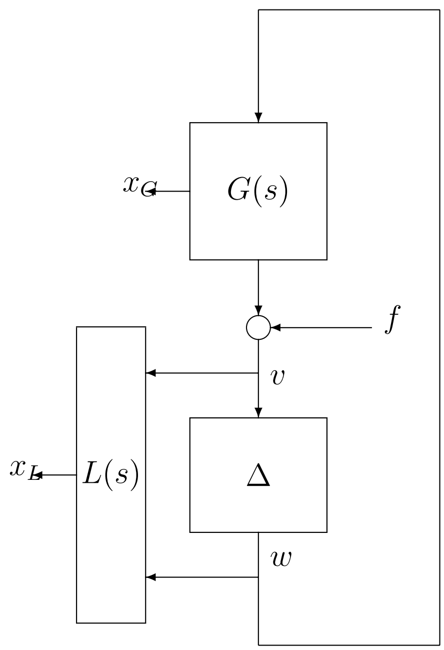

Figure 13.5:

where A_L, B_L^v, B_L^w are known real matrices.

Thus, the extended system model $\mathcal{S} = \{z\}$ will have

$$z = [v; w; x_G; x_L],$$

constrained by the known LTI relations (13.5),(13.7), and by the constraints imposed by Δ. One can consider an IQC model of $\mathcal{S}$ with

$$f = v - C_G x_G - D_G w, \quad y = v.$$

Note that, while the choice of f is rigidly dictated by the original setup, the definition of y is quite arbitrary here. One IQC for $\mathcal{S}$ is given by (13.6), which is considered an *abstraction* of Δ. On the other hand, the IQC for the linear relations (13.5),(13.7) are given by Theorem 13.3 in the form

$$\sigma_P(x_G, x_L, v, w) \geq \dot{V}_P,$$

where

$$\sigma_P(x_G, x_L, v, w) = 2 \begin{bmatrix} x_G \\ x_L \end{bmatrix}' P \begin{bmatrix} A_G x_G + B_G w \\ A_L x_L + B_L^v v + B_L^w w \end{bmatrix},$$

$$V_P(z,T) = \begin{bmatrix} x_G(T) \\ x_L(T) \end{bmatrix}' P \begin{bmatrix} x_G(T) \\ x_L(T) \end{bmatrix},$$

and $P = P'$ is an arbitrary symmetric real matrix of appropriate dimensions.

Feasibility analysis of the IQC model described here means finding a symmetric matrix P such that

$$\sigma_\Delta(x_L, v, w) + \sigma_P(x_G, x_L, v, w) \le \gamma |v - C_G x_G - D_G w|^2 - |v|^2$$

for all real vectors x_G, x_L, v, w, while minimizing the value of γ. According to the KYP Lemma (Theorem 13.4), it is sufficient to check that the inequality holds when x_G, x_L, v, w are complex vectors satisfying the equations

$$j\omega x_G = A_G x_G + B_G w, \quad j\omega x_L = A_L x_L + B_L^v v + B_L^w w, \tag{13.8}$$

for all $\omega \in \mathbf{R}$ (and, for at least one value of $\omega \in \mathbf{R}$, the inequality must hold strictly). Note that, subject to (13.8),

$$C_G x + D_G w = G(j\omega) w, \quad \sigma_P(x_G, x_L, v, w) = 0, \quad \sigma_\Delta(x_L, v, w) = \begin{bmatrix} v \\ w \end{bmatrix}' \Pi(j\omega) \begin{bmatrix} v \\ w \end{bmatrix},$$

where

$$\Pi(j\omega) = \begin{bmatrix} (j\omega I - A)^{-1}[B_L^v \ , \ B_L^w] \\ I \end{bmatrix}' \Sigma_\Delta \begin{bmatrix} (j\omega I - A)^{-1}[B_L^v \ , \ B_L^w] \\ I \end{bmatrix}.$$

Therefore the frequency domain condition of feasibility of the IQC model is

$$\Pi(j\omega) \le \gamma \begin{bmatrix} I \\ -G(j\omega)' \end{bmatrix} \begin{bmatrix} I \\ -G(j\omega)' \end{bmatrix}' - \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \quad \forall\, \omega \in \mathbf{R}.$$

To guarantee that the inequality holds for *some* (sufficiently large) $\gamma > 0$ it is sufficient to satisfy the simpler inequality

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}' \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \le -\epsilon I. \tag{13.9}$$

Note that matrix $\Pi = \Pi(j\omega)$ comes naturally from the original IQC for Δ: assuming that v and w are square integrable over $(0, \infty)$ yields

$$\int_0^\infty \sigma_\Delta(x_L, v, w) dt = \int_{-\infty}^\infty \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}' \Pi(j\omega) \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} d\omega.$$

One can give an interpretation of $\Pi(j\omega)$ as the weight matrix which describes the redistribution of energy in the spectrum of signals as v gets transformed into w by subsystem Δ. From this viewpoint, (13.9) is a natural generalization of the harmonic balance idea from the special case of the circle criterion, where

$$\Pi(j\omega) = \begin{bmatrix} I & 0 \\ 0 & -\gamma_0 I \end{bmatrix}.$$

13.4.4 Existence of Storage Functions

An important advantage of IQC analysis is the possibility of using an IQC $\sigma \geq \dot{V}$ without knowing an explicit expression for the storage function V. The following theorem allows one to derive existence of a storage function satisfying a certain IQC.

Let $\sigma : \mathbf{R}^m \to \mathbf{R}$ be a quadratic form. It is said that system $\mathcal{S} \subset L_{2,loc}^m$ satisfies the *conditional* IQC defined by σ if for every $z_0 \in \mathcal{S}$ and $T > 0$ the infimum

$$U(z_0, T) = \inf_{z \in \mathcal{F}(z_0,T)} \int_T^\infty \sigma(z(t))dt,$$

taken over the set $\mathcal{F}(z_0, T)$ of all signals from $\mathcal{S}$ which are square integrable and define same state of $\mathcal{S}$ as z_0 at time T, is finite.

Theorem 13.5 *If $\mathcal{S} \subset L_{2,loc}^m$ satisfies the conditional IQC defined by quadratic form $\sigma : \mathbf{R}^m \to \mathbf{R}$ then the IQC $\sigma \geq \dot{V}$ holds for some V.*

Proof The proof is similar to that of Theorem 13.3, defining $V(z, T) = -U(z, T)$. Then (13.2) is the standard Bellman inequality, and (13.1) follows from the observation that

$$U(z, T) \leq \int_T^\infty \sigma(z(t))dt \to 0 \quad \text{as} \quad T \to \infty$$

for all $z \in \mathcal{S} \cap L_2^m$. ■

13.5 Lower Bounds of Storage Functions

Consider an IQC model of a system $\mathcal{S} = \{z\} \subset L_{2,loc}^m$, with external input $f = L_{in}z$, transient response output $y = L_{out}z$, and a set of IQC

$$\sigma_k(z) \geq \dot{V}_k, \quad k = 1, 2, \ldots$$

Assume the IQC model is feasible, i.e. the quadratic form

$$\gamma|f|^2 - |y|^2 - \sum c_k\sigma_k(z) = \gamma|L_{in}z|^2 - |L_{out}z|^2 - \sum c_k\sigma_k(z)$$

is positive semidefinite for some choice of $\gamma, c_k \geq 0$. As it was pointed out before, IQC feasibility proves stability when the corresponding storage function

$$V = \sum c_k V_k$$

is non-negative, and proves instability otherwise. Thus, obtaining easy-to-handle lower bounds of the storage functions V_k is an important component of IQC analys.

13.5.1 Input-Output Setup for IQC

Let $\Delta = \{h = [v; w]\} \subset L_{2,loc}^{k+q}$ be a system describing a relation between vector signals $v = v(t) \in \mathbf{R}^k$ and $w = w(t) \in \mathbf{R}^q$. Let A, B_1, B_2 be real matrices of dimensions n-by-n, n-by-k, and n-by-q respectively, such that A is a Hurwitz matrix. Let X_0 be a subset in $\mathbf{R}^n$. Define $\mathcal{S} \subset L_{2,loc}^m$, where $m = k + q + n$, as the set of all $z = [v; w; x] \in L_{2,loc}^m$ such that $[v; w] \in \Delta$ and

$$\dot{x}(t) = Ax(t) + B_1v(t) + B_2w(t), \quad x(0) \in X_0. \tag{13.10}$$

Here Δ describes the "complex" component of a feedback system $\mathcal{S}$, while (13.10) represents a nominal LTI feedback, as well as the auxiliary signals introduced when deriving IQC for Δ. For example, (13.10) may combine equations (13.5),(13.7), in which case

$$x = [x_G; x_L], \quad X_0 = \{[x_G; x_L] : \ x_L = 0\}.$$

Note that two signals $z_1 = [v_1; w_1; x_1]$ and $z_2 = [v_2; w_2; x_2]$ define same state of $\mathcal{S}$ at time $T > 0$ if and only if $x_1(T) = x_2(T)$ and signals $[v_1; w_1]$, $[v_2; w_2]$ define same state of Δ at time T.

Assume that an IQC $\sigma(x, v, w) \geq \dot{V}$ is satisfied for $\mathcal{S}$, where σ is a known quadratic form on $\mathbf{R}^m$, while the storage function $V : \ \mathcal{S} \times (0, \infty) \to \mathbf{R}$ is not known. The following question is of major interest: given a symmetric n-by-n matrix $Q = Q'$ (where n is the dimension of $x(t)$) when is it possible to claim that the inequality $V(z,T) \geq x(T)'Qx(T)$ holds for all $z = [v; w; x] \in \mathcal{S}$, $T > 0$?

To derive such quadratic lower bounds for V, some assumptions of *causality* and *stability* are to be imposed on the relation between v and w defined by Δ. The basic inequality

$$V(z,T) \geq -\int_T^\infty \sigma(z_0(t))dt, \tag{13.11}$$

which holds for every $z_0 \in \mathcal{S} \cap L_2^m$ defining same state of $\mathcal{S}$ at time T as z, is to be used to get a lower bound for $V(z,T)$. More precisely, for $z = [v; w; x] \in \mathcal{S}$, $z_0 = [v_0; w_0; x_0]$ will be defined by a signal $[v_0; w_0] \in \Delta$ such that $v_0(t) = v(t)$, $w_0(t) = w(t)$ for $t < T$, for which the integral of $\sigma(z_0(t))$ from T to infinity is small. Depending on the situation, z_0 is either explicitly assumed to exist, or can be proven to exist using some indirect assumptions.

13.5.2 Zero In - Zero Out

The following assumption about Δ leads to a simple lower bound for V:

(**ZIZO**): for every $h = [v; w] \in \Delta$ and $T > 0$ there exists $h_0 = [v_0; w_0] \in \Delta \cap L_2^{k+q}$ such that h and h_0 define same state of Δ at time T, $h(t) = h_0(t)$ for all $t < T$, and $h_0(t) = 0$ for all $t > T$.

In other words, (**ZIZO**) means that v can be switched to zero at any time $T > 0$, and this will make w zero as well. The assumption holds for unbiased memoryless relations between v and w, such that $w(t) = t|v(t)|$, though it can also be true for some nonlinear dynamical relations, such as $w(t) = v(t-1)\sin(v(t))$.

Theorem 13.6 *Assuming (***ZIZO***),*

$$V(z,T) \geq x(T)'Qx(T),$$

where

$$QA + A'Q = Q_0, \ \ \sigma([0;0;a]) = a'Q_0a \ \forall \ a \in \mathbf{R}^n.$$

Note that $\sigma([0;0;a]) \leq 0$ would imply $Q \geq 0$ and hence $V(z,T) \geq 0$.

Proof For z_0 defined by $[v_0; w_0]$ from (**A0**) we have

$$x_0(T+\tau) = e^{A\tau}x_0(T) \ \ \forall \ \tau \geq 0.$$

Hence (13.11) yields

$$V(z,T) \geq -\int_T^\infty \sigma([0;0;x_0(t)]) = -\int_0^\infty (e^{At}x(T))'Q_0(e^{At}x(T))dt = x(T)'Qx(T).$$

■

It is easy to see that the conclusion of Theorem 13.6 still holds when (**ZIZO**) is relaxed by assuming existence of extensions h_0 with an arbitrarily small integral of $|h_0(t)|^2$ over (T,∞).

13.5.3 Convex IQC

The following assumption about Δ is much less restrictive:

(**SC**): for every $h=[v;w]\in\Delta$, $T>0$, and $d\in L_2^k$ there exists $h_0=[v_0;w_0]\in\Delta\cap L_2^{k+q}$ such that h and h_0 define same state of Δ at time T, $h(t)=h_0(t)$ for all $t<T$, and $v_0(t)=d(t)$ for all $t>T$.

In other words, (**SC**) means that v can be switched to any other square integrable signal at any time $T>0$, and this will make w square integrable as well. The assumption holds for most stable causal transformations from v to w.

When working with (**SC**), we will need an additional assumption about σ, A, B_2: the existence of $\epsilon>0$ such that

$$\sigma([0;W;X]) \leq -\epsilon|W|^2 \quad \forall\ X\in\mathbf{C}^n, W\in\mathbf{C}^q, \omega\in\mathbf{R}:\ j\omega X = AX+B_2W. \tag{13.12}$$

Condition (13.12) means strict convexity of the integral

$$J(w)=\int_0^\infty \sigma([0;w(t);x(t)])dt:\ \dot{x}=Ax+B_2w,\ x(0)=0$$

as a function of $w\in L_2^q$, i.e. in some sense strict convexity of the IQC $\sigma([v;w;x])\geq\dot{V}$ as a constraint imposed on w.

Theorem 13.7 *Assuming (*__SC__*) and (13.12), there exists n-by-n matrix* $Q=Q'$ *such that*

$$J_0(a) = -a'Qa = \inf_{v\in L_2^k}\max_{w\in L_2^q}\int_0^\infty \sigma([v;w;x])dt:\ \dot{x}=Ax+B_1v+B_2w,\ x(0)=a$$

for all $a\in\mathbf{R}^n$ *which belong to the set*

$$X_{\mathcal{S}} = \{x(T):\ [v;w;x]\in\mathcal{S},\ T>0\},$$

and the inequality

$$V(z,T) \geq x(T)'Qx(T),$$

holds for all $z \in \mathcal{S}, T > 0$.

Proof We will only give a sketch of the proof here. On one hand, (13.12) guarantees that the maximum

$$J_1(a,v) = \max_{w \in L_2^q} \int_0^\infty \sigma([v;w;x])dt : \ \dot{x} = Ax + B_1 v + B_2 w, \ x(0) = a$$

exists and is finite for all $v \in L_2^k$, $a \in \mathbf{R}^n$. By (13.11) we have

$$V(z,T) \geq -J_1(x(T),v)$$

for every $z \in \mathcal{S}$, $T > 0$. Hence $J_0(a)$ is finite for all $a \in X_{\mathcal{S}}$, and $V(z,T) \geq -J_0(x(T))$. Since maximum (or an infimum) of a quadratic form over all perturbations of the argument along a given linear subspace is a quadratic form again, $J_1(a,v)$ and $J_0(a)$ must be quadratic forms. ■

It is possible to produce an algorithm for calculating matrix Q from Theorem 13.7 using Riccati equation solvers and standard linear equation solvers. This is left as an excercise for the reader familiar with linear-quadratic optimization and KYP lemma.

It is easy to see that the conclusion of Theorem 13.7 still holds when (**SC**) is relaxed by assuming existence of extensions h_0 for which the integral of $|v_0(t) - d(t)|^2$ over (T, ∞) can be made arbitrarily small.

13.5.4 A Minimax Condition for $V \geq 0$

The result presented in this subsection utilizes Theorem 13.7 to show that under a very non-restrictive assumption feasibility of a convex IQC model implies non-negativity of the corresponding storage function, and hence stability of the original system.

Theorem 13.8 *Assume that conditions (***SC***) and (13.12) are satisfied. In addition, assume that there exists* $\delta > 0$ *such that*

$$\sigma([V;0;X]) \geq \delta |V|^2 \ \ \forall \ X \in \mathbf{C}^n, V \in \mathbf{C}^q, \omega \in \mathbf{R} : \ j\omega X = AX + B_1 V. \qquad (13.13)$$

If

$$\sigma([Cx + Dw; w; x]) \leq 0 \ \ \forall \ w \in \mathbf{R}^q, x \in \mathbf{R}^n, \qquad (13.14)$$

and $A + B_1 C$ *is a Hurwitz matrix then* $V(z,T) \geq 0$.

Note that the last two assumptions of Theorem 13.8 are usually satisfied automatically when $v - Cx - Dw = f$ plays the role of an external signal, and $Cx + Dw$ represents the response of the nominal system model to input w. A mor essential assumption is given by (13.13), which essentially says that the *zero* model of Δ, i.e. the one defined by

$$\Delta_0 = \{[v; w] \in L_{2,loc}^k \times L_{2,loc}^q : \ w(t) \equiv 0\}$$

satisfies the IQC defined by $\sigma - \delta|v|^2$ (i.e. in a *strict* sense), though with a (possibly) different storage function. Since the IQC are frequently used to describe the difference between the nominal and the true dynamics, such assumption about the IQC appears to be reasonable.

Proof Returning to the proof of Theorem 13.7, note that the integral

$$J_2(a, v, w) = \int_0^\infty \sigma([v; w; x])dt : \ \dot{x} = Ax + B_1 v + B_2 w, \ x(0) = a$$

is strictly convex with respect to w and strictly concave with respect to v. Hence, by the minimax theorem, we have

$$-a'Qa = \min_{v \in L_2^k} \max_{w \in L_2^q} J_2(a, v, w) = \max_{w \in L_2^q} \min_{v \in L_2^k} J_2(a, v, w).$$

However, for a fixed $w \in L_2^q$, the stability assumption about $A + B_1 C$ allows one to choose $v \in L_2^k$ to satisfy $v = Cx + Dw$. Then, by (13.14), $J_2(a, v, w) \leq 0$, which implies $Q \geq 0$. ■

13.6 Examples

Practical verification of *strict* feasibility of IQC models can be done by standard routines of convex optimization. Furthermore, the IQC's known for a number of standard nonlinear, uncertain, or time-varying components, such as nonlinearities with conic, slope, and curvature bounds, LTI uncertainty (real parametric, unmodeled, repeated, uncertain delay), time-varying uncertain gain (fast and slowly time-varying, harmonic, periodic), and many others, can be pre-stored in a computer for easy access. In this way, rather complicated IQC models can be built and tested for feasibility in a very simple manner.

13.6.1 Example: a servo with friction and uncertain delay

Consider a simple model $\mathcal{S}_{\text{servo}}$ of a servo with friction on Figure 13.6, where

$$K(s) = 10\frac{2s^2+2s+1}{0.01s^2+s+0.01} \approx 20s+20+10/s$$

represents a PID controller, the saturated high gain feedback models friction, and $\tau \in [0, 0.05]$ is the uncertain measurement delay. In terms of signals, u is control input, f is

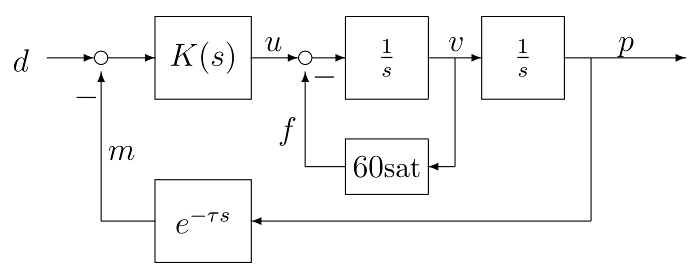

Figure 13.6: Example: a servo with friction

the friction force, v and p are velocity and position while m is a position measurement with uncertain delay. The external input d represents a high-pass sensor noise with the cut-off at 100 rad/sec. The objective is to prove stability of the system and to estimate the noise amplification coefficient J, defined as the maximal L_2 gain in the closed loop channel $e \mapsto p$, where $\dot{d} + 100d = \dot{e}$.

A feasibility test for the IQC model can be specified in the toolbox `iqc_beta` with the commands[2]

[2]This code will only work with the MATLAB version 5.3 or (possibly) higher

```
s=tf([1 0],1);  % convenient constant
abst_init_iqc;  % initialize data structure for handling IQC
e=signal; % external unmodeled disturbance
f=signal; % a nonlinear gain output
r=signal; % an uncertain block output
p=signal; % position (feedback loop signal)
m=p+r;    % delayed position
u=(10*(2*s^2+2*s+1)/(0.01*s^2+s+0.01))*((s/(s+100))*e-m);
v=(1/s)*(u-f); % velocity
p==(1/s)*v;  % closing the loop
f==60*iqc_sector(v); % f(f-60u)>=0
r==iqc_ltvnorm(p)-p; % ||r+p||^2<=||p||^2
J=iqc_gain_tbx(e,p) % estimate L2 gain   e -> p
```

Here the command `abst_init_iqc` initializes the data structure, and `==` is used to close feedback loops. Expression `iqc_sector(v)` produces a signal z_1 that satisfies the IQC defined by the quadratic form $z_1(z_1 - v)$. The essential part of the code of `iqc_sector` can be written as

```
function w=iqc_sector(v)
w=signal;   % define the output
c=symmetric; % introduce a scaling constant
c>0;  % put an LMI constraint on c
w'*c*(w-v)<0; % describe an IQC constraint
```

Similarly, the expression `iqc_ltvnorm(p)` produces a signal z_2 that satisfies the IQC defined by the quadratic form $|z_2|^2 - |p|^2$. Finally, the command `J=iqc_gain_tbx(e,p)` indicates that $f = e$ is the external disturbance, and $y = p$ is the transient response to be considered.

Running the script produces an empty J, which means that no $\sigma \in \tilde{\Sigma}$ such that $\sigma < \sigma_0$ was found. This does not necessarily imply that the original servo system is unstable – just that the IQC model is not good enough.

A more accurate IQC model of the servo system would use the (additional) IQC. To check feasibility of the upgraded IQC model, the MATLAB code above should be modified by replacing `iqc_sector(v)` and `iqc_ltvnorm(p)-p` with `iqc_monotonic(v)` and `iqc_cdelay(p,.05)` respectively. Running the modified script produces $J \approx 58$, which means that the new IQC model is strictly feasible for γ as small as 3600. Invoking an appropriate IQC post-feasibility result (for example, Theorem 13.8) shows that the resulting storage function V is non-negative, and hence the servo system is stable, and the noise amplification coefficient J does not exceed 60.

13.6.2 Example with cubic nonlinearity and delay

For an application of IQC analysis where strict feasibility does not take place consider the following system of differential equations[3] with an uncertain constant delay parameter τ:

$$\dot{x}_1(t) = -x_1(t)^3 - x_2(t-\tau)^3 \qquad (13.15)$$

$$\dot{x}_2(t) = x_1(t) - x_2(t) \qquad (13.16)$$

Analysis of this system is easy when $\tau = 0$, and becomes more difficult when τ is an arbitrary constant in the interval $[0, \tau_0]$. The system is not exponentially stable for any value of τ. Our objective is to show that, despite the absence of exponential stability, the method of IQC can be applied.

For $\tau = 0$, we begin with describing (13.15),(13.16) by the behavior set

$$\mathcal{Z} = \{z = [x_1; x_2; w_1; w_2]\},$$

where

$$w_1 = x_1^3,\ w_2 = x_2^3,\ \dot{x}_1 = -w_1 - w_2,\ \dot{x}_2 = x_1 - x_2.$$

The *trivial* IQC for $\mathcal{Z}$ are given by

$$\sigma_{LTI}(z) = 2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' P \begin{bmatrix} -w_1 - w_2 \\ x_1 - x_2 \end{bmatrix},$$

where $P = P'$ is an arbitrary symmetric 2-by-2 matrix. Among the non-trivial IQC's valid for $\mathcal{Z}$, the simplest two represent the circle and the Popov criteria, and are defined by

$$\sigma_{NL}(z) = d_1 x_1 w_1 + d_2 x_2 w_2 + q_1 w_1(-w_1 - w_2) + q_2 w_2 (x_1 - x_2),$$

$$V_\sigma(z(\cdot), t) = 0.25(q_1 x_1(t)^4 + q_2 x_2(t)^4),$$

where $d_k \geq 0$. Let $\tilde{\Sigma}$ be the cone of matrices of the quadratic forms σ. Since we are only proving stability, let $\sigma_0 = 0$. It turns out (and is easy to verify) that the only solutions of the IQC feasibility problem $\sigma \leq 0$ are the ones that make $\sigma = \sigma_{LTI} + \sigma_{NL} = 0$, for example

$$P = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}, \qquad d_1 = d_2 = q_2 = 1, \qquad q_1 = 0.$$

The absence of strictly feasible solutions corresponds to the fact that the system is not exponentially stable. Nevertheless, a Lyapunov function candidate can be constructed from the given solution:

$$V(x) = x'Px + 0.25(q_1 x_1^4 + q_2 x_2^4) = 0.5 x_1^2 + 0.25 x_2^4.$$

[3]Suggested by Petar Kokotovich

This Lyapunov function can be used along the standard lines to prove global asymptotic stability of the equilibrium $x = 0$ in system (13.15),(13.16).

Now consider the case when $\tau \in [0, 0.2]$ is an uncertain parameter. To show that the delayed system (13.15),(13.16) remains stable when $\tau \leq 0.2$, (13.15),(13.16) can be represented by a more elaborate behavior set $\mathcal{Z} = \{z(\cdot)\}$ with

$$z = [x_1; x_2; w_1; w_2; w_3; w_4; w_5; w_6] \in \mathbf{R}^8,$$

satisfying LTI relations

$$\dot{x}_1 = -w_1 - w_2 + w_3, \quad \dot{x}_2 = x_1 - x_2$$

and the nonlinear/infinite dimensional relations

$$w_1(t) = x_1^3,\ w_2 = x_2^3,\ w_3 = x_2^3 - (x_2 + w_4)^3,$$

$$w_4(t) = x_2(t - \tau) - x_2(t),\ w_5 = w_4^3,\ w_6 = (x_1 - x_2)^3.$$

Some additional IQC are needed to bound the new variables. These will be selected using the perspective of a small gain argument. Note that the perturbation w_4 can easily be bounded in terms of $\dot{x}_2 = x_1 - x_2$. In fact, the LTI system with transfer function $(\exp(-\tau s) - 1)/s$ has a small gain (in almost any sense) when τ is small. Hence a small gain argument would be applicable provided that the gain "from w_4 to $\dot{x}_2$" could be bounded as well.

It turns out that the Λ_2-induced gain from w_4 to $\dot{x}_2$ is unbounded. Instead, we can use the Λ_4 norms. Indeed, the last two components w_5, w_6 of w were introduced in order to handle L_4 norms within the framework of IQC. More specifically, in addition to the trivial IQC with

$$\sigma_{LTI}(z) = 2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' P \begin{bmatrix} -w_1 - w_2 + w_3 \\ x_1 - x_2 \end{bmatrix},$$

the set $\mathcal{Z}$ satisfies the IQC $\sigma \geq \dot{V}$, where

$$\begin{aligned} \sigma(z) = & d_1 x_1 w_1 + d_2 x_2 w_2 + q_1 w_1(-w_1 - w_2 + w_3) + q_2 w_2(x_1 - x_2) \\ & + d_3[0.99(x_1 w_1 + x_2 w_2) - x_1 w_3 + 2.5^4 w_4 w_5 - 0.5^4 (x_1 - x_2) w_6] \\ & + q_3[0.2^4 (x_1 - x_2) w_6 - w_4 w_5], \end{aligned}$$

$d_i \geq 0$. Here the IQC with coefficients d_1, d_2, q_1, q_2 are same as before. The term with d_3, based on a zero storage function, follows from the inequality

$$0.99(x_1^4 + x_2^4) - x_1(x_2^3 - (x_2 + w_4)^3) + \left(\frac{5w_4}{2}\right)^4 - \left(\frac{x_1 - x_2}{2}\right)^4 \geq 0$$

(which is satisfied for *all* real numbers x_1, x_2, w_4, and can be checked numerically).

The term with q_3 follows from a gain bound on the transfer function $G_\tau(s) = (\exp(-\tau s) - 1)/s$ from $x_1 - x_2$ to w_4. It is easy to verify that the Λ_1 norm of its impulse response equals τ, and hence the L^4 induced gain of the causal LTI system with transfer function G_τ will not exceed 1. Consider the function

$$V_d(v(\cdot), T) = -\inf \int_T^\infty \left\{ 0.2^4 |v_1(t)|^4 - \left| \int_{t-\tau}^t v_1(r) dr \right|^4 \right\} dt, \qquad (13.17)$$

where the infimum is taken over all functions v_1 which are square integrable on $(0, \infty)$ and such that $v_1(t) = v(t)$ for $t \leq T$. Because of the L_4 gain bound of G_τ with $\tau \in [0, 0.2]$ does not exceed 0.2, the infimum in (13.17) is bounded. Since we can always use $v_1(t) = 0$ for $t > T$, the infimum is non-positive, and hence V_d is non-negative. The IQC defined by the "q_3" term holds with $V_\sigma = q_3 V_d(x_1 - x_2, t)$.

Let

$$\sigma_0(z) = -0.01(x_1 w_1 + x_2 w_2) = -0.01(x_1^4 + x_2^4),$$

which reflects our intention to show that x_1, x_2 will be integrable with fourth power over $(0, \infty)$.

The IQC model cannot be made strictly feasible, but is feasible for

$$P = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}, \quad d_1 = d_2 = 0.01, \ d_3 = q_2 = 1, \ q_1 = 0, \ q_3 = 2.5^4.$$

A Lyapunov function candidate can be constructed with the help of these P, d_k, q_k:

$$V(x_e(t)) = 0.5 x_1(t)^2 + 0.25 x_2(t)^4 + 2.5^4 V_d(x_1 - x_2, t),$$

where x_e is the "total state" of the system (in this case, $x_e(T) = [x(T); v_T(\cdot)]$, where $v_T(\cdot) \in L_2(0, \tau)$ denotes the signal $v(t) = x_1(T - \tau + t) - x_2(T - \tau + t)$ restricted to the interval $t \in (0, \tau)$). From the solution of the IQC feasibility problem, it follows that

$$\frac{dV(x_e(t))}{dt} \leq -0.01(x_1(t)^4 + x_2(t)^4).$$

On the other hand, we saw previously that $V(x_e(t)) \geq 0$ is bounded from below. Therefore, $x_1(\cdot), x_2(\cdot) \in \Lambda_4$ (fourth powers of x_1, x_2 are integrable over $(0, \infty)$) as long as the initial conditions are bounded. Thus, the equilibrium $x = 0$ in system (13.15),(13.16) is stable for $0 \leq \tau \leq 0.2$.