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6.245: MULTIVARIABLE CONTROL SYSTEMS

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The Tustin Transform¹

So far, the optimization methods under consideration were presented for the continuous time case. This lecture describes a technique which allows one to apply the continuous time algorithms of H₂ optimization, H-Infinity optimization, and Hankel optimal model reduction to discrete time systems. The technique is based on applying the familiar “Tustin” (or “bilinear”) CT to DT transformation. While the standard systems software in MATLAB frequently solves DT problems by transforming them into CT format, solving the resulting CT problem, and then transforming the solution back, in some situations it may be beneficial to solve a CT problem by transforming it first into a DT format.

11.1 Properties of the Tustin Transform

This section introduces the Tustin transform for transfer matrices and state space models, and describes some useful properties of the transform.

11.1.1 Tustin transform for transfer matrices

For a discrete time (DT) transfer function $H = H(z)$, its *Tustin transform at frequency* $\omega_0 > 0$ is defined by

$$G(s) = \text{Tu}_{\omega_0}[H](s) = H\left(\frac{\omega_0 + s}{\omega_0 - s}\right). \quad (11.1)$$

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The inverse of Tu_{ω_0} takes a CT system $G(s)$ and produces the DT system with transfer matrix

$$H(z) = \text{Tu}_{\omega_0}^{-1}[G](s) = G\left(\omega_0 \frac{z-1}{z+1}\right). \quad (11.2)$$

For example, applying the Tustin transform at frequency $\omega_0 = 1$ to $H(z) = 1/z$ yields $G(s) = (1-s)/(1+s)$.

The Tustin transform converts a number of properties of CT transfer matrices into similar properties of DT transfer matrices. This set of equivalencies is summarized by the following statement.

Theorem 11.1 *$G = G(s)$ is a Tustin transform of a rational transfer matrix $H = H(z)$ if and only if G is rational and H is the inverse Tustin transform of G . Moreover, in this case*

- (a) *the CT L-Infinity norm of $G = G(s)$ equals the DT L-Infinity norm of $H = H(z)$, i.e.*

$$\sup_{\text{Re}(s)=0} \sigma_{\max}(G(s)) = \sup_{|z|=1} \sigma_{\max}(H(z));$$

- (b) *G is CT stable (bounded in the region $\text{Re}(s) > 0$) if and only if H is DT stable (bounded in the region $|z| > 1$);*

- (c) *if H is strictly proper and DT stable, and ω_0 is the frequency of the Tustin transform, then $G(\omega_0) = 0$,*

$$\tilde{G}(s) = \frac{\sqrt{2\omega_0}}{s - \omega_0} G(s)$$

is strictly proper and CT stable, and the CT H2 norm of \tilde{G} equals the DT H2 norm of H , defined by

$$\|H\|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \|H(\exp(j\tau))\|_F^2 d\tau.$$

Proof Since, for

$$z = \frac{\omega_0 + s}{\omega_0 - s}, \quad s = \omega_0 \frac{z-1}{z+1},$$

condition $|z| = 1$ is equivalent to $s \in j\mathbf{R} \cup \{\infty\}$, equality in (a) follows from the definitions. In a similar way, (b) follows from the equivalence of $|z| > 1$ and $\text{Re}(s) > 0$. Finally, to prove (c), note that

$$\omega_0 \frac{e^{jt} - 1}{e^{jt} + 1} \equiv j\omega$$

implies

$$dt = \frac{2\omega_0}{\omega^2 + \omega_0^2} d\omega,$$

hence

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| G \left(\omega_0 \frac{e^{jt} - 1}{e^{jt} + 1} \right) \right|^2 dt = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|G(j\omega)|^2}{\omega^2 + \omega_0^2} d\omega.$$

Since H is strictly proper, $H(z) = 0$ at $z = \infty$, which means that $G(s) = 0$ at $s = \omega_0$. Hence $\tilde{G}(s) = \sqrt{2\omega_0}G(s)/(s - \omega_0)$ is CT stable, and the CT H2 norm of \tilde{G} equals the DT H2 norm of H . ■

11.1.2 Tustin transform for state space models

Let $H = H(z)$ be a transfer matrix with a state space model

$$x[k+1] = ax[k] + bf[k], \quad g[k] = cx[k] + df[k],$$

where f is the input and g is the output. In general, the Tustin transform G of H does not have to be a proper transfer matrix. However, this will be the case when $z = -1$ is not a pole of H . Hence, if -1 is not an eigenvalue of a , there exists an explicit expression for a state space model of G .

Theorem 11.2 (a) *If -1 is not an eigenvalue of a then the Tustin transform (at frequency ω_0) $G = G(s)$ of*

$$H(z) = d + c(zI - a)^{-1}b \tag{11.3}$$

is given by

$$G(s) = \bar{d} + \bar{c}(sI - \bar{a})^{-1}\bar{b}, \tag{11.4}$$

where

$$\bar{d} = d - c(I + a)^{-1}b, \quad \bar{c} = \sqrt{2\omega_0}c(I + a)^{-1}, \quad \bar{b} = \sqrt{2\omega_0}(I + a)^{-1}b, \quad \bar{a} = \omega_0(a - I)(a + I)^{-1}, \tag{11.5}$$

and ω_0 is not an eigenvalue of \bar{a} .

(b) *If ω_0 is not an eigenvalue of \bar{a} then the Tustin transform (at frequency ω_0) $H = H(z)$ of the CT transfer matrix (11.4) is given by (11.3), where*

$$d = \bar{d} - \bar{c}(\omega_0 I - \bar{a})^{-1}\bar{b}, \quad c = \sqrt{2\omega_0}\bar{c}(\omega_0 I - \bar{a})^{-1}, \quad b = \sqrt{2\omega_0}(\omega_0 I - \bar{a})^{-1}\bar{b}, \quad a = (\omega_0 I + \bar{a})(\omega_0 I - \bar{a})^{-1},$$

and -1 is not an eigenvalue of a .

The proof of Theorem 11.2 follows by inspection. Note that the pair (a, b) is controllable if and only if the pair (\bar{a}, \bar{b}) is controllable. Similarly, (c, a) is observable if and only if (\bar{c}, \bar{a}) is observable. Therefore, as long as -1 is not an eigenvalue of a and ω_0 is not an eigenvalue of \bar{a} , the order of H equals the order of G .

11.2 Optimization via Tustin transform

The general scheme of using Tustin transform in optimization of discrete time systems is quite straightforward: apply the Tustin transform to the original DT setup, do the CT optimization, then apply inverse Tustin transform to get the solution in the original DT problem. However, since the Tustin transform maps some proper transfer matrices into transfer matrices which are not proper, and since the H2 norm is not preserved under the transform, there is a number of technical issues which need careful consideration.

11.2.1 Hankel optimal model reduction

Let us define Hankel norm of a given DT stable (i.e. bounded in the region $|z| > 1$) rational transfer matrix $H(z)$ as the minimum of the DT L-Infinity norm of $H + H_-$, taken over all rational transfer matrices H_- which are anti-stable (i.e. bounded in the region $|z| < 1$). Consequently, the DT Hankel optimal model reduction problem is defined as the task of finding, for a given DT stable transfer matrix $H = H(z)$, a DT stable transfer matrix \hat{H} of order less than a given positive integer m , such that the Hankel norm of $H - \hat{H}$ is minimal.

The discrete time Hankel optimal model reduction problem is an example of a problem for which application of the Tustin transform goes without complications.

Theorem 11.3 *If $G(s)$ is Tustin transform of a DT stable transfer matrix H then G is also stable, and the CT Hankel norm of G equals the DT Hankel norm of H . Similarly, if $H(z)$ is inverse Tustin transform of a CT stable transfer matrix $G = G(s)$ then H is also stable, and the DT Hankel norm of H equals the CT Hankel norm of G . Moreover, if $\hat{G}(s)$ is a CT Hankel optimal reduced model of G of order k then its inverse Tustin transform $\hat{H}(z)$ is the Hankel optimal reduced model of H of order k .*

The proof of Theorem 11.3 follows easily from the observation (following from the state space formulae for the Tustin transform) that order, stability, and anti-stability is preserved under the Tustin transform for rational DT transfer matrices bounded on the unit circle, and, symmetrically, these properties are preserved under the inverse Tustin transform for rational CT transfer matrices bounded on the imaginary axis.

11.2.2 The standard DT LTI feedback design setup

The standard DT LTI feedback design setup is defined by a DT state space model in which the input is partitioned into disturbance w and control u , and the output is partitioned into cost z and sensor y :

$$x[k+1] = Ax[k] + B_1w[k] + B_2u[k], \quad (11.6)$$

$$z[k] = C_1x[k] + D_{11}w[k] + D_{12}u[k], \quad (11.7)$$

$$y[k] = C_2x[k] + D_{21}w[k] + D_{22}u[k]. \quad (11.8)$$

Without loss of generality, we can assume that $D_{22} = 0$ and a stabilizing *proper* feedback system

$$x_f[k+1] = A_fx_f[k] + B_fy[k], \quad (11.9)$$

$$u[k] = C_fx_f[k] + D_fy[k], \quad (11.10)$$

is to be designed, where the controller is called stabilizing if the “ A ” matrix A_{cl} of the closed loop system is a Schur matrix (all poles have absolute value less than one). Alternatively, one can formulate the task as that of designing a *strictly proper* controller (11.9),(11.10), in which case D_{22} can be defined arbitrarily.

As in the CT case, either H2 or H-Infinity DT norm of the closed loop transfer matrix from w to z is minimized.

Note that, for a stabilizing controller to exist, the pair (A, B_2) must be DT stabilizable (i.e. $A + B_2F$ is a Schur matrix for some F) and the pair (C_2, A) must be DT detectable (i.e. $A + LC$ is a Schur matrix for some L). In addition, the standard DT setup is said to have a *control singularity* at a point z on the unit circle (i.e. $|z| = 1$) if the matrix

$$M_u(z) = \begin{bmatrix} A - zI & B_2 \\ C_1 & D_{12} \end{bmatrix}$$

is not left invertible. Similarly, the standard DT setup is said to have a *sensor singularity* at a point z on the unit circle if the matrix

$$M_y(z) = \begin{bmatrix} A - zI & B_1 \\ C_2 & D_{21} \end{bmatrix}$$

is not right invertible. Note that, unlike the CT case, singularity at $z = \infty$ is not a concern, and, accordingly, matrices D_{12}, D_{21} could be zero without making the setup singular.

11.2.3 H-Infinity optimization via Tustin transform

For convenience, we will consider the case when $D_{22} = 0$ and a general proper stabilizing controller is to be designed to minimize the closed loop DT H-Infinity norm of the transfer function from w to z .

Well posedness of a standard DT LTI feedback design setup does not always guarantee that Tustin transform can be applied directly to the state space model (11.6)-(11.8), because matrix $I + A$ is not guaranteed to be invertible. However, a standard setup (11.6)-(11.8) can be replaced by an equivalent optimization problem by re-defining the control variable according to

$$u^{new} = u - K_0 y.$$

In terms of the new control variable, system coefficients change according to

$$\begin{aligned} A^{new} &= A + B_2 K_0 C_2, & B_1^{new} &= B_1 + B_2 K_0 D_{21}, & B_2^{new} &= B_2, \\ C_1^{new} &= C_1 + D_{12} K_0 C_2, & D_{11}^{new} &= D_{11} + D_{12} K_0 D_{21}, & D_{12}^{new} &= D_{12}, \\ C_2^{new} &= C_2, & D_{21}^{new} &= D_{21}. \end{aligned}$$

It is easy to verify by inspection that well the new setup is well posed (non-singular and output feedback stabilizable) if and only if the original one is. The important claim is that, for an appropriate selection of the constant matrix K_0 , $I + A^{new} = I + A + B_2 K_0 C_2$ will be invertible.

Theorem 11.4 *If real matrices a, b, c (where a is a square matrix) are such that $a + bf$ and $a + hc$ are invertible for some real matrices f, c then $a + bkc$ is invertible for some real matrix k .*

Proof Note that the assumptions of the theorem, as well as existence of k such that $a + bkc$ is invertible, do not change under the “coordinate transformation”

$$a \mapsto v_1 a v_2, \quad b \mapsto v_1 b v_3, \quad c \mapsto v_4 c v_2,$$

where v_1, v_2, v_3, v_4 are invertible square matrices, as well as under the “feedback transformation”

$$a \mapsto a + b k_f c, \quad b \mapsto b, \quad c \mapsto c.$$

By applying an appropriate coordinate transformation, one can have a, b, c of the compatible block form

$$a = \begin{bmatrix} I & 0 & a_{13} \\ 0 & 0 & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad b = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ I & 0 \end{bmatrix}, \quad c = \begin{bmatrix} 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix},$$

where some of the block dimensions could be zero. Applying an appropriate feedback transformation to these matrices yields a similar setup with $a_{13} = 0$ and $a_{31} = 0$. Now, by the assumption of invertibility of $a + hc$ for some h , the rows of a_{23} must be linearly independent. Similarly, by the assumption of invertibility of $a + bf$ for some f , the columns of a_{23} must be linearly independent. Hence, after an appropriate coordinate transformation, we can get

$$a_{32} = \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad a_{23} = [I \quad 0].$$

In this system of coordinates, defining

$$k = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} - a_{33}$$

yields

$$a + bkc = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}.$$

■

For a well posed setup, Theorem 11.4 can be applied to

$$a = I + A, \quad b = B_2, \quad c = C_2$$

to prove (constructively) the existence of a matrix K_0 such that $I + A + B_2 K_0 C_2$ is invertible. Hence, after the output feedback transformation $u^{new} = u - K_0 y$, the Tustin transform at frequency ω_0 can be applied to the setup. Using the state space formulae for Tustin transform, it can be verified easily that the resulting standard CT feedback design setup will be well posed. In addition, since the DT transfer matrix from u to y is strictly proper, the CT plant transfer matrix from u to y equals zero at $s = \omega_0$ (remember that $s = \omega_0$ will *not* be an open loop CT pole). Let $K^{CT} = K^{CT}(s)$ define a stabilizing feedback for the CT plant achieving the closed loop H-Infinity norm γ . Then ω_0 is not a pole of K^{CT} , and hence a Tustin transform can be applied to K^{CT} to obtain a proper stabilizing DT feedback transfer matrix $K = K(z)$ achieving the closed loop H-Infinity norm γ .

11.2.4 H2 optimization via Tustin transform

For discrete time H2 optimization, we will consider a setup in which a strictly proper controller is being designed. It will also be assumed that $I + A$ is invertible (otherwise,

an output feedback transformation of the control variable should be applied, as described in the previous subsection, to make $I + A$ invertible). Since for a stable DT system H we have

$$\|H\|_2^2 = \|H(\infty)\|_F^2 + \|H - H(\infty)\|_2^2,$$

and, when using a strictly proper controller, the closed loop gain from w to z at $z = \infty$ does not depend on the controller (actually, the gain equals D_{11}), we can assume that $D_{11} = 0$. Finally, D_{22} will be chosen in such way that the open loop transfer matrix from u to y equals zero at $z = -1$.

Under these assumptions, the Tustin transform $P = P(s)$ of the original DT plant will exist, and will satisfy the conditions

$$P_{11}(\omega_0) = 0, \quad P_{22}(\infty) = 0.$$

For a proper stabilizing controller $K^c = K^c(s)$, the loop transfer function T_{wz} is given by

$$\frac{1}{s - \omega_0} T_{wz} = \tilde{P}_{11} + P_{12} \tilde{K} (I - \tilde{P}_{22} \tilde{K})^{-1} P_{21},$$

where

$$\tilde{K}(s) = \frac{K^c(s)}{s - \omega_0}, \quad \tilde{P}_{11}(s) = \frac{P_{11}(s)}{s - \omega_0}, \quad \tilde{P}_{22}(s) = (s - \omega_0) P_{22}(s).$$

Let \tilde{K}_* be the strictly proper optimal controller in the H2 feedback optimization problem defined by the plant

$$\tilde{P} = \begin{bmatrix} \tilde{P}_{11} & P_{12} \\ P_{21} \tilde{P}_{22} & \end{bmatrix}.$$

Since $\tilde{P}_{22}(\omega_0) = 0$, \tilde{K}_* does not have a pole at $s = \omega_0$. Hence the inverse Tustin transform can be applied to

$$K_*(s) = (s - \omega_0) \tilde{K}_*(s)$$

to obtain a strictly proper DT controller $K_* = K_*(z)$. By construction, this controller will be optimal in the original DT H2 optimal feedback design problem.