Massachusetts Institute of Technology

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6.245: MULTIVARIABLE CONTROL SYSTEMS

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Problem Set 8 Solutions [1]

Problem 8.1

For the standard LTI feedback design setup defined by equations

$$\dot{x}(t) = ax(t) + u(t) + w_1(t),\ z(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix},\ y = \begin{bmatrix} \dot{x}(t) \\ x(t) + w_2(t) \end{bmatrix},$$

where $a \in \mathbf{R}$ is a parameter, find matrices T_0, T_1, T_2 defining a valid Q-parameterization of all closed loop transfer matrices $T: \ w \to z$ which can be achieved while using a finite order stabilizing dynamic feedback $u = Ky$.

This state space model has matrix coefficients

$$A = a,\ B_1 = \begin{bmatrix} 1 & 0 \end{bmatrix},\ B_2 = 1,\ C_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ C_2 = \begin{bmatrix} a \\ 1 \end{bmatrix},$$

$$D_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\ D_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ D_{21} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ D_{22} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Since $D_{22} \neq 0$, consider the case when K is strictly proper. Then the set of all achievable closed loop transfer matrices will not change when D_{22} is replaced by zero. Indeed, for a strictly proper $K = K(s)$,

$$u = K(C_2x + D_{21}w + D_{22}u)$$

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is equivalent to

$$u = \hat{K}(C_2 x + D_{21} w),$$

where

$$\hat{K} = (I - K D_{22})^{-1} K, \quad K = \hat{K}(I + \hat{K} D_{22})^{-1},$$

and $\hat{K}$ is strictly proper whenever K is strictly proper.

To apply the Q-parameterization theorem of Lecture 10, take F, L such that $A + B_2 F$ and $A + LC_2$ are Hurwitz matrices, for example,

$$F = -1 - a, \quad L = [-1 \quad -1],$$

which yields

$$A + B_2 F = A + LC_2 = -1.$$

Then the explicit formulae for the transfer matrices T_0, T_1, T_2 in

$$T_{wz} = T_0 + T_1 Q T_2, \quad Q \ - \ \text{stable, proper}$$

yield

$$T_0(s) = \begin{bmatrix} \frac{1}{s+1} & -\frac{1+a}{(s+1)^2} \\ -\frac{1+a}{s+1} & -\frac{1+a}{s+1} + \frac{(1+a)^2}{(s+1)^2} \end{bmatrix}, \; T_1(s) = \begin{bmatrix} \frac{1}{s+1} \\ 1 - \frac{1+a}{s+1} \end{bmatrix}, \; T_2(s) = \begin{bmatrix} 1 & -\frac{a}{s+1} \\ 0 & \frac{s}{s+1} \end{bmatrix}.$$

Problem 8.2

For the standard discrete time LTI feedback design setup defined by equations

$$x[k+1] = -x[k] + u[k] + w_1[k], \;\; z[k] = \begin{bmatrix} ax[k] \\ u[k] \end{bmatrix}, \; y[k] = x[k] + w_2[k],$$

where $a > 0$ is a parameter, find the H2 optimal feedbacl law by using a Tustin transformation to an equivalent continuous time problem. Also give explicit expressions for the equivalent CT setup, and for the corresponing CT H2 optimal feedback.

As in most discrete time formulations, this H2 optimization setuo has at least two different interpretations: the one in which only strictly causal controllers are allowed (i.e. $u[k]$ is allowed to depend on $y[k-1], y[k-1], \dots$), and the one with an arbitrary causal controller ($u[k]$ depends on $y[k], y[k-1], \dots$). Below, a solution for the second case is presented.

First, since there is an open loop pole at $z=-1$, we introduce the new control variable

$$v[k] = u[k] - y[k] = u[k] - x[k] - w_2[k].$$

The new system equations will have the form

$$x[k+1] = w_1[k] + w_2[k] + v[k],\ \ z[k] = \begin{bmatrix} ax[k] \\ x[k] + w_2[k] + v[k] \end{bmatrix},\ y[k] = x[k] + w_2[k].$$

Then, in order to reduce the problem to that of desining a strictly causal controller, introduce the new sensor output

$$\bar{y}[k] = y[k+1] = x[k+1] + w_2[k+1] = w_1[k] + w_2[k] + v[k] + w_2[k+1].$$

Since the equations now depend on both $w_2[k]$ and $w_2[k+1]$, we introduce an additional system state $x_2[k] = w_2[k]$, and a modified noise vector

$$f[k] = \begin{bmatrix} f_1[k] \\ f_2[k] \end{bmatrix} = \begin{bmatrix} w_1[k] \\ w_2[k+1] \end{bmatrix}.$$

Now system equations have the form

$$\begin{bmatrix} x_1[k+1] \\ x_2[k+1] \end{bmatrix} = \begin{bmatrix} x_2[k] + f_1[k] + v[k] \\ f_2[k] \end{bmatrix},\ \ z[k] = \begin{bmatrix} ax_1[k] \\ x_1[k] + x_2[k] + v[k] \end{bmatrix},\ \bar{y}[k] = x_2[k]+f_1[k]+f_2[k]+v[k],$$

where $x_1[k] = x[k]$. Here $D_{11} = 0$. To make sure that the transfer matrix from control to sensor is zero at $\mathbf{z} = -1$, use $D_{22} = 0$, which means introducing a new sensor variable

$$y_*[k] = \bar{y}[k] - v[k] = x_2[k] + f_1[k] + f_2[k].$$

Now the open loop plant transfer matrix is

$$P^{DT}(z) = \begin{bmatrix} \frac{a}{z} & \frac{a}{z^2} & \frac{a}{z} \\ \frac{1}{z} & \frac{z+1}{z^2} & \frac{z+1}{z} \\ 1 & \frac{z+1}{z} & 0 \end{bmatrix}.$$

Applyin the Tustin transform

$$z = \frac{1+s}{1-s},\quad s = \frac{z-1}{z+1},$$

yields a continuous time plant

$$P(s) = \begin{bmatrix} a\frac{1-s}{1+s} & a\frac{(1-s)^2}{(1+s)^2} & a\frac{1-s}{1+s} \\ \frac{1-s}{1+s} & 2\frac{1-s}{(1+s)^2} & \frac{2}{1+s} \\ 1 & \frac{2}{1+s} & 0 \end{bmatrix}.$$

After dividing $P_{fz}(s)$ by $1-s$ and multiplying $P_{vy}(s)$ by $1-s$, we get

$$\hat{P}(s) = \begin{bmatrix} \frac{a}{1+s} & \frac{a(1-s)}{(1+s)^2} & a\frac{1-s}{1+s} \\ \frac{1}{1+s} & \frac{2}{(1+s)^2} & \frac{2}{1+s} \\ 1 & \frac{2}{1+s} & 0 \end{bmatrix}.$$

A state space model of this CT plant is given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -x_1(t) + 2x_2(t) + f_1(t) + 2v(t) \\ -x_2(t) + f_2(t) \end{bmatrix}, \; z(t) = \begin{bmatrix} a(x_1(t) - x_2(t) - v(t)) \\ x_1(t) \end{bmatrix}, \; y(t) = f_1(t) + 2x_2(t).$$

To solve the corresponding standard CT H2 optimization problem, consider the associated full information abstract H2 optimization:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_1 + 2x_2 + 2u \\ -x_2 \end{bmatrix}, \; \int_0^\infty \{|x_1|^2 + a^2|u + x_2 - x_1|^2\}dt \to \min.$$

With a modified control vector

$$\tilde{u} = u + x_2 - x_1,$$

the problem can be re-written in an equivalent one-dimensional form

$$\dot{x}_1 = x_1 + 2\tilde{u}, \quad \int_0^\infty \{|x_1|^2 + a^2|\tilde{u}|^2\}dt \to \min.$$

The optimal controller is given by

$$u = -g_f x_1 - x_2,$$

where

$$g_f = \sqrt{\frac{1}{4} + \frac{1}{a^2}} - \frac{1}{2} = 1 - \frac{2p_f}{a^2},$$

and

$$p_f = \frac{a^2}{4} + \sqrt{\frac{a^2}{4} + \frac{a^4}{16}}$$

is the stabilizing solution of the associated Riccati equation

$$p^2 - \frac{a^2}{2}p - \frac{a^2}{4} = 0.$$

Similarly, consider the associated state estimation abstract H2 optimization:

$$\begin{bmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \end{bmatrix} = \begin{bmatrix} -\psi_1 \\ 2\psi_1 - \psi_2 + 2q \end{bmatrix}, \int_0^\infty \{|\psi_1 + q|^2 + |\psi_2|^2\} dt \rightarrow \min .$$

With a modified control vector

$$\tilde{q} = \psi_1 + q,$$

the problem can be re-written in an equivalent one-dimensional form

$$\dot{\psi}_2 = -\psi_2 + 2\tilde{q}, \quad \int_0^\infty \{|\psi_2|^2 + |\tilde{q}|^2\} dt \rightarrow \min .$$

The optimal controller is given by

$$q = -\psi_1 - g_e \psi_2,$$

where

$$g_e = \sqrt{\frac{1}{4} + 1} - \frac{1}{2} = 2p_e,$$

and

$$p_e = -\frac{1}{4} + \sqrt{\frac{1}{4} + \frac{1}{16}}$$

is the stabilizing solution of the associated Riccati equation

$$p^2 + \frac{1}{2}p - \frac{1}{4} = 0.$$

The optimal CT controller K^{CT} has state space model

$$\frac{d}{dt} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} -\hat{x}_1 + 2\hat{x}_2 + 2u \\ -\hat{x}_2 \end{bmatrix} - \begin{bmatrix} 1 \\ g_e \end{bmatrix} (2\hat{x}_2 - y), \; u = -g_f \hat{x}_1 - \hat{x}_2,$$

which corresponds to controller transfer function

$$K^{CT}(s) = -\frac{g_f(s+1) + g_e(s+1+2g_f)}{(s+1+2g_f)(s+1+2g_e)}.$$

The inverse Tustin transform $\hat{K}(z)$ of $(1-s)K^{CT}(s)$ is given by

$$\hat{K}(z) = -\frac{2zg_f + g_e(2z + g_f(z+1))}{(2z + g_f(z+1))(2z + g_e(z+1))}.$$

Due to the changes of control and sensor variables introduced in the original setup, the true optimal controller (with input $y[k]$ and output $u[k]$) is given by

$$K(z) = 1 + \frac{z\hat{K}(z)}{1+\hat{K}(z)},$$

which yields

$$K(z) = \frac{4z - g_e g_f}{(2+g_e)(2+g_f)z + g_e g_f}.$$

Problem 8.3

Consider a system described by the hyperbolic partial differential equation

$$v_t = v_{xx} + rv, \quad v(0,t) = 0, \quad y(t) = v(1,t) + w(t), \; u(t) = v_x(1,t),$$

where $v = v(x,t)$, for fixed time, is a function of the spatial parameter $x \in [0,1]$, v_t denotes the time derivative of v, v_{xx} denotes the double spatial derivative of v, and $r > 0$ is a given parameter. The control action is the Dirichlet boundary condition $u(t) = v_x(1,t)$, while a noisy measurement of $y(t) = v(1,t) + w(t)$ is used as the sensor signal.

(a) Find an analytical expression for the transfer function $P = P_r(s)$ from u to y.

(b) For $r = 1$, find a good low order rational approximation $\hat{P}_1$ of P_1, such that $\Delta = P_1 - \hat{P}_1$ is stable, together with an upper bound $\|\Delta\|_\infty < \epsilon$.

(c) Using the results from (b), small gain theorem, and H-Infinity optimization, design a finite order stabilizing feedback $u = Ky$ for the original system, while trying to provide an upper bound for the closed loop H-Infinity norm $\|T_{wu}\|$ which is as small as possible. Note that this will only be possible when ϵ is small enough.