Massachusetts Institute of Technology

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6.245: MULTIVARIABLE CONTROL SYSTEMS

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Problem Set 6 Solutions [1]

Problem 6.1

A standard feedback control design setup is defined by the differential equations

$$\dot{x}_1 = -ax_1 + u, \quad \dot{x}_2 = -x_2 + w,$$

and by

$$y = w + bx_2, \ z = \begin{bmatrix} cu \\ x_1 \end{bmatrix},$$

where a, b, c are real parameters.

(a) Find matrices $A, B_1, B_2, C_1, C_2, D_{11}, D_{12}, D_{21}, D_{22}$ for this setup.

By inspection,

$$A = \begin{bmatrix} -a & 0 \\ 0 & -1 \end{bmatrix}, \ B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ C_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \ C_2 = \begin{bmatrix} 0 & b \end{bmatrix},$$

$$D_{11} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ D_{12} = \begin{bmatrix} c \\ 0 \end{bmatrix}, \ D_{21} = 1, \ D_{22} = 0.$$

(b) Find all values of a, b, c for which the setup is singular, inficating frequency, multiplicity, and type (control/sensor) of the singularity.

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[1] Version of April 26, 2004

Matrix

$$M_u(s) = \begin{bmatrix} -a-s & 0 & 1 \\ 0 & -1-s & 0 \\ 0 & 0 & c \\ 1 & 0 & 0 \end{bmatrix}$$

is right invertible for all $s = j\omega$, $\omega \in \mathbf{R}$. The largest degree of determinant of a 3-by-3 minor of $M_u(s)$ equals $d_u^\infty = 1$ when $c = 0$, and 2 otherwise. Hence, for $c = 0$, there is a control singularity at $\omega = \infty$, multiplicity 1 (the difference between system order and d_u^∞). For $c \neq 0$, there are no control singularities.

The determinant of

$$M_y(s) = \begin{bmatrix} -a-s & 0 & 0 \\ 0 & -1-s & 1 \\ 0 & b & 1 \end{bmatrix}$$

equals

$$\delta_y(s) = \det(M_y(s)) = (s+a)(s+b+1).$$

Hence, the system has sensor singularities only when $a = 0$ or $b = -1$, in which case the singularity is located at $\omega = 0$, and its multiplicity 1 equals the multiplicity of the zero $s = 0$ of $\delta_y(s)$ (one unless $a = b + 1 = 0$, which implies multiplicity 2).

Problem 6.2

FIND H-INFINITY AND H2 NORMS OF $G(s) = G_a(s)$ AS A FUNCTION OF REAL PARAMETER a:

(a) $G(s) = 1/(s+a) - 2/(s+2a)$;

The impulse response $g = g(t)$ of G is given by

$$g(t) = e^{-at} - 2e^{-2at}.$$

Since

$$\int_0^\infty |g(t)|^2 dt = \int_0^\infty \left\{e^{-2at} - 4e^{-3at} + 4e^{-4at}\right\} dt = \frac{1}{6a},$$

$\|G\|_{H2} = 1/\sqrt{6a}$.

Since

$$G(s) = -\frac{s}{s^2 + 3as + 2a^2} = -\frac{1}{3a + s + 2a^2/s},$$

we have $|G(j\omega)| \geq 1/3a$, where the equality takes place at $\omega = \sqrt{2}a$. Hence $\|G\|_\infty = 1/3a$.

(b) $G(s) = (1 - \exp(-a^2 s))/s$.

The impulse response $g = g(t)$ of G is given by

$$g(t) = \begin{cases} 1, & t \in [0, a^2], \\ 0, & \text{otherwise.} \end{cases}$$

Hence $\|G\|_{H2} = a$, and $\|G\|_\infty = a^2$.

Problem 6.3

Find L2 gains of systems described below (input f, output g, defined for $t \geq 0$). You are not required to *prove* correctness of your answer.

(a) $y(t) = e^{-at} f(t)$.

L2 gain equals 1 for $a \geq 0$, and infinity for $a < 0$.

(b) $y(t) = f(t)/(1 + a^2 f(t)^2)$.

L2 gain equals 1 for all a.

(c) $y(t) = f(t + a)$.

L2 gain equals 1 for $a \leq 0$, and infinity for $a > 0$.

Problem 6.4

A standard H2 optimization setup is defined by the following transfer functions:

$$P_{wz}(s) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \;\; P_{wy}(s) = 1, \;\; P_{uz}(s) = \begin{bmatrix} 1/(s+a) \\ 1 \end{bmatrix}, \;\; P_{uy}(s) = -1/(s+a),$$

where $a \in \mathbf{R}$ is a parameter. For all values of a for which the setup is non-singular, find the H2 optimal controller, together with the associated Hamiltonian matrices, solutions of Riccati equations, and controller/observer gains.

A state space model of the setup is given by

$$\dot{x} = -ax + u, \;\; x(0) = x_0, \;\; z = \begin{bmatrix} x \\ u \end{bmatrix}, \;\; y = w - x.$$

The setup is non-singular for $a \neq 0$ (when $a = 0$, there is a sensor singularity at $\omega = 0$).

The full information abstract H2 optimization has the form

$$\dot{X} = -aX + U, \quad \int_0^\infty (|X|^2 + |U|^2)dt \to \min .$$

The corresponding Hamiltonian matrix is

$$\mathcal{H}_{fi} = \begin{bmatrix} -a & 1 \\ 1 & a \end{bmatrix}.$$

The stabilizing solution of the associated Riccati equation is

$$P_{fi} = \sqrt{a^2+1} - a,$$

and the optimal full state feedback gain is given by

$$K_{fi} = a - \sqrt{a^2+1}.$$

The state estimation abstract H2 optimization has the form

$$\dot{\psi} = -a\psi - q, \ \psi(0) = \psi_0, \quad \int_0^\infty |q|^2 dt \to \min .$$

The corresponding Hamiltonian matrix is

$$\mathcal{H}_{se} = \begin{bmatrix} -a & 1 \\ 0 & a \end{bmatrix}.$$

The stabilizing solution of the associated Riccati equation is

$$P_{se} = \begin{cases} 0, & \text{for} \quad a > 0, \\ -2a, & \text{for} \quad a < 0, \end{cases}$$

and the optimal state estimator gain is given by

$$L_{se} = \begin{cases} 0, & \text{for} \quad a > 0, \\ -2a, & \text{for} \quad a < 0. \end{cases}$$

For $a > 0$, the H2 optimal controller has zero transfer function. For $a < 0$, the H2 optimal controller can be written in the observer-based form

$$u = (a - \sqrt{a^2+1})\hat{x}, \ \frac{d}{dt}\hat{x} = -a\hat{x} + u - 2a(-\hat{x} - y),$$

and its transfer function is

$$K(s) = \frac{2a(a - \sqrt{a^2+1})}{s + \sqrt{a^2+1} - 2a}.$$

Problem 6.5

Open-loop plant $P(s)$ is strictly proper, has a double pole at $s = 0$, and a zero at $s = 0.1$. Controller $C = C(s)$ stabilizes the system on Figure 6.1,

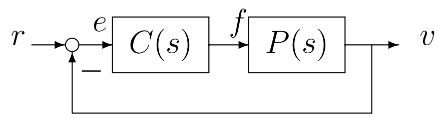

Figure 6.1: A SISO Feedback Setup

and ensures good tracking (transfer function from r to e has gain less than 0.1) for frequencies up to 1 rad/sec. Find a good lower bound on the maximal gain from r to e.

Let S denote the closed loop transfer function from r to e. Then, by assumption, $S(0.1) = 1$. Hence

$$\int_0^{\infty} \frac{\log |S(j\omega)| d\omega}{\omega^2 + 0.1^2} \geq 0.$$

Using the fact that $|S(j\omega)| \leq 0.1$ for $\omega \leq 1$, and $|S(j\omega)| \leq \|S\|_\infty$ for $\omega \geq 1$, we get

$$\log \|S\|_\infty \int_{10}^{\infty} \frac{d\omega}{\omega^2 + 1} \geq \log 10 \int_0^{10} \frac{d\omega}{\omega^2 + 1},$$

i.e.

$$\|S\|_\infty \geq 10^{\frac{\arctan(10)}{\pi/2 - \arctan(10)}} \geq 10^{14.76}.$$

Problem 6.6

Find the square of the H2 norm of

$$G(s) = \frac{s}{s^2 + a_1 s + a_0}$$

as a rational function of real parameters a_1, a_0.

A state space model of G is given by

$$A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = 0.$$

The solution P of the Lyapunov equation

$$AP + PA' = -BB'$$

has the form

$$P = \begin{bmatrix} 1/2a_0a_1 & 0 \\ 0 & 1/2a_1 \end{bmatrix}.$$

Hence

$$\|G\|_{H2} = \sqrt{CPC'} = \frac{1}{\sqrt{2a_1}}.$$