Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.245: MULTIVARIABLE CONTROL SYSTEMS

by A. Megretski

Problem Set 5 Solutions [1]

Problem 5.1

Use KYP Lemma to find (analytically) the set of all $a \in \mathbf{R}$ such that the Riccati equation

$$PA + A'P = (C' - PB)(C - B'P),$$

where (A, B) is controllable, (C, A) is observable, and

$$C(sI - A)^{-1}B = (s + a)^{-1000},$$

has a stabilizing solution $P = P'$.

This is a Riccati equation of the form

$$\alpha + P\beta + \beta' P = P\gamma P,$$

where

$$\alpha = -C'C, \ \beta = A + BC, \ \gamma = BB'.$$

Since the pair (A, B) is controllable, so is the pair $(A + BC, B)$. According to the KYP Lemma, a stabilizing solution of the Riccati equation exists if and only if

$$|w|^2 - |Cx|^2 \gg 0 \ \text{ for } \ j\omega x = (A + BC)x + Bw, \ \omega \in \mathbf{R}.$$

Substitution $v = w + Cx$ yields an equivalent condition

$$|v|^2 - 2\text{Re}(v'Cx) \gg 0 \ \text{ for } \ j\omega x = Ax + Bv, \ \omega \in \mathbf{R}.$$

[1] Version of April 26, 2004

Again, according to the KYP lemma,this is equivalent to

$$1 > \text{Re } G(j\omega) \ \ \forall \ \omega \in \mathbf{R}, \ G(s) = C(sI - A)^{-1}B.$$

Since the maximal real part of $G(j\omega)$ is achieved at $\omega = 0$, a stabilizing solution of the Riccati equation exists if and only if $|a| > 2^{0.001}$.

Problem 5.2

Using the generalized Parrot's theorem, write down an algorithm for finding matrix L which minimizes the largest eigenvalue of

$$M = M(L) = \left[\begin{array}{cc} \alpha & \beta + 2L \\ 2L' + \beta' & \gamma + L'L \end{array} \right],$$

where $\alpha = \alpha'$, β, and $\gamma = \gamma'$ are given matrices.

First, let us find the lower bound λ_* for the functional to be minimized. Note that $\lambda_{\max}(M(L)) < r$ if nd only if the quadratic form

$$\sigma_r(w, u, y) = w'\alpha w + 2\text{Re } w'(\beta y + 2u) + y'\gamma y + |u|^2 - r(|w|^2 + |y|^2)$$

is negative definite for $u = Ly$. Conditions for existence of such L are given by the generalized Parrot's theorem (which can be applied because σ_r is convex with respect to u):

(a) $\alpha < rI$ (i.e. $\sigma_r(w, 0, 0) \ll 0$);

(b) $\left[\begin{array}{cc} \alpha - 4I & b \\ \beta' & \gamma \end{array} \right] < rI$ (i.e. the minimum of $\sigma_r(w, u, y)$ with respect to u is negative definite).

Hence

$$\lambda_* = \max \left\{ \lambda_{\max}(\alpha), \lambda_{\max} \left(\left[\begin{array}{cc} \alpha - 4I & b \\ \beta' & \gamma \end{array} \right] \right) \right\}.$$

Now, for $r = \lambda_*$, let $u_* = c_1 y + c_2 w$ be the argument of minimum of $\sigma_r(w, u, y)$ with respect to u (it is easy to see that, in our case, $c_1 = 0$ and $c_2 = -2$). Let

$$\sigma_r^*(w, y) = \sigma_r(w, c_1 y + c_2 w, y) = w'\alpha w + 2\text{Re } w'\beta y + y'\gamma y - 4|w|^2 - \lambda_*(|w|^2 + |y|^2)$$

be the minimum itself. Let $w_* = c_3 y$ be the argument of maximum of $\sigma_r^*(w, y)$ with respect to w (since $\alpha < \lambda_* I$, $\sigma_r^*(w, y)$ is strictly concave with respect to w, hence a unique maximum is well defined). It is easy to see that, in our case,

$$c_3 = (4I + \lambda_* I - \alpha)^{-1}\beta.$$

To complete a solution, let us prove that

$$L = L_* = c_1 + c_2 c_3 = -2(4I + \lambda_* I - \alpha)^{-1}\beta$$

is an optimal value of L. Indeed, according to the way c_1, c_2, c_3 are defined,

$$\sigma_r(w,u,y) = |u - c_1 y - c_2 w|^2 - (w - c_3 y)'(4I + \lambda_* I - \alpha)(w - c_3 y) + \sigma_r^{**}(y),$$

where

$$\sigma_r^{**}(y) = \max_w \sigma_r^*(w,y) = \max_w \min_u \sigma_r(w,u,y) = y'(\gamma - \lambda_* I + \beta'(4I + \lambda_* I - \alpha)^{-1}\beta)y \le 0.$$

When $u = (c_1 + c_2 c_3)y$, we have

$$\sigma_r(w,u,y) = |c_2(w - c_3 y)|^2 - (w - c_3 y)'(4I + \lambda_* I - \alpha)(w - c_3 y) + \sigma_r^{**}(y) = w'(\alpha - \lambda_* I)w + \sigma_r^{**}(y) \le 0.$$

Problem 5.3

USE THE KYP LEMMA TO WRITE A MATLAB ALGORITHM FOR CHECKING THAT A GIVEN STABLE TRANSFER FUNCTION $G = G(s)$, AVAILABLE IN A STATE SPACE FORM, SATISFIES THE CONDITION

$$|G(j\omega)| > 1 \quad \forall\ \omega \in \mathbf{R} \cup \{\infty\}.$$

THE ALGORITHM SHOULD BE EXACT, PROVIDED THAT THE LINEAR ALGEBRA OPERATIONS INVOLVED (MATRIX MULTIPLICATIONS, EIGENVALUE CALCULATIONS, COMPARISON OF REAL NUMBERS) ARE PERFORMED WITHOUT NUMERICAL ERRORS. IN PARTICULAR, CHECKING THAT $|G(j\omega_k)| > 1$ AT A FINITE SET OF FREQUENCIES ω_k IS NOT ACCEPTABLE IN THIS PROBLEM[2].

Assume that a minimal state space model of G is given by

$$G := \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right).$$

Note that condition $|G(j\omega)|^2 > 1$ is equivalent to

$$|Cx + Dw|^2 - |w|^2$$

being positive definite subject to $j\omega x = Ax + Bw$ for all real ω, including $\omega = \infty$, in which case the linear constraint takes the form $x = 0$. According to the KYP Lemma,

[2] Of course, frequency sampling may be acceptable in many practical applications

this is equivalent to the inequality $D'D > I$ plus the existence of a stabilizing solution $P = P'$ of the Riccati equation

$$\alpha + P\beta + \beta' P = P\gamma P,$$

where

$$\alpha = C'(I - D(D'D - I)^{-1}D')C,\ \beta = A - B(D'D - I)^{-1}D'C,\ \gamma = B(D'D - I)^{-1}B'.$$

The second condition is equivalent to the absence of purely imaginary eigenvalues of the associated Hamiltonian matrix

$$\mathcal{H} = \begin{bmatrix} \beta & \gamma \\ \alpha & -\beta \end{bmatrix}.$$

The M-function `ps5_3.m` implements the algorithm. When its argument d is less than one, it either reports the "D condition" $D'D > I$ is not satisfied, or produces a very small (numerically indistinguishable from zero) minimal absolute value of the real part of eigenvalues of the associated Hamiltonian matrix.