Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.245: MULTIVARIABLE CONTROL SYSTEMS

by A. Megretski

Problem Set 4 Solution [1]

Problem 4.1

For the SISO feedback design from Figure 4.1, where it is known that $P(2) = 0$ and $1 \pm 2j$ are poles of P, find a lower bound (as good as you can) on the H-Infinity norm of the closed-loop complementary sensitivity transfer function $T = T(s)$ (from r to v), assuming that $C = C(s)$ is a stabilizing controller, $|T(j\omega) - 1| < 0.2$ for $|\omega| < 10$, and $|T(j\omega)| < 0.1$ for $|\omega| > 20$.

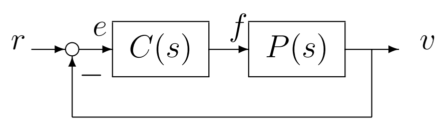

Figure 4.1: A SISO Feedback Setup

Since $P(2) = 0$ and $P(1 \pm 2j) = \infty$, we have $S(2) = 1$ and $S(1 \pm 2j) = 0$, where $S = 1 - T$ is the sensitivity function. By specifications, $|S(j\omega)| < 0.2$ for $|\omega| < 10$, and $|S(j\omega)| < 1.1$ for $|\omega| > 20$. Let $z_1, \dots, z_n$ denote the unstable zeros (possibly repeated) of S, except the ones at $1 \pm 2j$. Define

$$S_{mp} = \frac{s^2 + 2s + 5}{s^2 - 2s + 5} \frac{s + z_1}{s - z_1} \dots \frac{s + z_n}{s - z_n}.$$

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Then S_{mp} is stable, has no unstable zeros, and satisfies $S(\infty) = 1$. Hence $\log(S_{mp})$ belongs to the class H2, and

$$\log |S_{mp}(2)| = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{2\log|S_{mp}(j\omega)|d\omega}{4+\omega^2} = \frac{2}{\pi}\int_{0}^{\infty} \frac{\log|S(2j\omega)|d\omega}{1+\omega^2}.$$

Since $|S_{mp}(j\omega)| = |S(j\omega)|$, and $S_{mp}(2) = 13/5$, this implies

$$\begin{aligned}\frac{\pi}{2}\log\frac{13}{5} &\le \int_0^5 \frac{\log(0.2)d\omega}{1+\omega^2} + \int_5^{10} \frac{\log(\|S\|_\infty)d\omega}{1+\omega^2} + \frac{\log(1.1)d\omega}{1+\omega^2} \\ &= \arctan(5)\log(0.2) + (\arctan(10)-\arctan(5))\log(\|S\|_\infty) + (\pi/2 - \arctan(10))\log(1.1).\end{aligned}$$

Hence

$$\log\|S\|_\infty \ge \frac{\frac{\pi}{2}\log\frac{13}{5} - \arctan(5)\log(0.2) - (\pi/2-\arctan(10))\log(1.1)}{\arctan(10)-\arctan(5)},$$

i.e.

$$\|T\|_\infty \ge \|S\|_\infty - 1 \ge 2.8\cdot 10^{16}.$$

Problem 4.2

(a) Apply the formulae for H2 optimization to a standard setup with a Hurwitz matrix A and with $B_2 = 0$ to express the H2 norm of CT LTI MIMO state space model

$$y = Cx, \ \dot{x} = Ax + Bf$$

in terms of matrices C and P, where $P = P'$ is the solution of the Lyapunov equation

$$AP + PA' = -BB'.$$

Consider the standard feedback optimization problem defined by

$$\dot{x} = Ax + Bw_1, \ z = \begin{bmatrix} Cx \\ u \end{bmatrix}, \ y = w_2, \ w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

The optimal controller is obviously $u \equiv 0$, i.e. the closed loop system has same H2 norm as $G(s) = C(sI - A)^{-1}B$.

According to the solution to H2 feedback optimization problem, the square of optimal H2 norm equals

$$\mathrm{trace}(B_1' P_{fi} B_1) + \mathrm{trace}(D_{12} K_{fi} P_{se} K_{fi}' D_{12}'),$$

where P_{fi}, P_{se} and K_{fi}, K_{se} are the stabilizing solutions of the Riccati equations and the corresponding optimal state feedback gains in the associated abstract H2 optimization problems.

In our case $K = L = 0$, and P_{fi}, P_{se} satisfy

$$C'C + P_{fi}A + A'P_{fi} = 0, \quad BB' + AP_{se} + P_{se}A' = 0.$$

This immediately yields the formula

$$\|G\|_{H2}^2 = \mathrm{trace}(B'W_oB),$$

where $W_o = P_{fi} = Q$ is the *observability Gammian* of system G, a solution of the Lyapunov equation

$$QA + A'Q = -C'C.$$

Noting that H2 norms of systems $C(sI - A)^{-1}B$ and $B'(sI - A')^{-1}C'$ must be equal (since trace of MM' equals trace of $M'M$ for all M), we get the equivalent formula

$$\|G\|_{H2}^2 = \mathrm{trace}(CW_cC'),$$

where $W_c = P_{se} = P$ is the *controllability Gammian* of system G, a solution of the Lyapunov equation

$$AP + PA' = -BB'.$$

(b) Use the result from (a) to obtain an explicit (with respect to a_0, a_1, a_2) formula for the H2 norm of system with transfer function

$$G(s) = \frac{1}{s^3 + a_2 s^2 + a_1 s + a_0},$$

where a_0, a_1, a_2 are positive real numbers such that $a_1 a_2 > a_0$.

Using the state space realization with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$

and solving the Lyapunov equation $AP + PA' = -BB'$ as a set of linear equations with respect to the components of $P = P'$, yields

$$P = \frac{1}{2(a_1a_2 - a_0)} \begin{bmatrix} a_2/a_0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & a_1 \end{bmatrix}.$$

Hence

$$\|G\|_{H2} = \sqrt{\frac{a_2}{2a_0(a_1a_2 - a_0)}}.$$

(c) Use MATLAB to check numerically correctness of your analytical solution.

The script is given in `ps4_2.m`.

Problem 4.3

An undamped linear oscillator with 3 degrees of freedom is described as a SISO LTI system with control input v, "position" output q, and transfer function

$$P_0(s) = \frac{1}{(1+s^2)(4+s^2)(16+s^2)}.$$

The position output q is measured with delay and noise, so that the sensor output g is defined as $g = 0.1f_1 + \hat{q}$, where $\hat{q}$ is output of an LTI system with input q and transfer function

$$P_1(s) = \frac{1-s}{1+s}.$$

The task is to design an LTI controller which takes g and a scalar reference signal r as inputs, produces v as its output, and satisfies the following specifications, assuming r is the output of an LTI system with input f_2 and transfer function

$$P_2(s) = \frac{\sqrt{2a}}{s+a},$$

where $a > 0$ is a parameter, and $f = [f_1; f_2]$ is a normalized white noise:

(A) the closed loop system is stable;

(B) the closed loop DC gain from r to q equals 1;

(c) the mean square value of control signal v is within 100;

(d) the mean square value of tracking error $e = q - r$ is as small as possible (within 20 percent of its minimum subject to constraints (a)-(c)).

Use H2 optimization to solve the problem for $a = 0.2$ and $a = 0.01$.

The MATLAB design code `ps4_3.m` uses SIMULINK diagram `ps4_3a.mdl`. Note how asymptotic tracking is forced by adding a pure integrator to the controller structure. Parameter r us used to tune the mean square control value (the smaller r, the large it is). Parameter d weights the artificial costs and noises introduces to assure non-singularity of the design setup. The Bode plot demonstrates asymptotic tracking in the closed loop system.

For $a = 0.01$, a decent tracking error of 0.2 (20 percent of reference) can be achieved. For $a = 0.2$ performance is very poor at 0.78. For $a = 1$, the “tracking” task is essentially not approached.