Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.245: MULTIVARIABLE CONTROL SYSTEMS

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Problem Set 3 Solutions [1]

Problem 3.1

Consider a control system described by

$$\ddot{q}(t) - a^2 q(t) = v(t) + f_1(t), \quad g(t) = q(t) + f_2(t),$$

where $f = [f_1; f_2]$ is a normalized white noise, v is the control signal, g is the sensor measurement, and $a > 0$ is a parameter. The objective is to find a dynamic feedback controller (with input g and output v) which stabilizes the system while using a minimum of control effort (defined as the asymptotic variance of $v(t)$ as $t \to \infty$).

(a) Find the coefficients of the auxiliary abstract H2 optimization problems associated with the original task.

For

$$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}, \; w = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \; u = v, \; y = g, \; z = v,$$

we have

$$A = \begin{bmatrix} 0 & 1 \\ a^2 & 0 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 & 0 \end{bmatrix}, \; C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \; D_{11} = \begin{bmatrix} 0 & 0 \end{bmatrix},$$

[1] Version of March 9, 2004

$$D_{12} = 1,\ D_{21} = \begin{bmatrix} 0 & 1 \end{bmatrix},\ D_{22} = 0.$$

The full information control abstract H2 optimization has coefficients

$$a = A,\ b = B_2,\ c = C_1,\ d = D_{12}.$$

The state estimation abstract H2 optimization problem has coefficients

$$a = A',\ b = C_2',\ c = B_1',\ d = D_{21}'.$$

(b) WRITE ANALYTICALLY THE ASSOCIATED HAMILTONIAN MATRICES, BASES OF THEIR STABLE INVARIANT SUBSPACES, STABILIZING SOLUTIONS OF THE RICCATI EQUATIONS, AND OPTIMAL CONTROLLER AND OBSERVER GAINS.

The full information control Hamiltonian is

$$\mathcal{H}_{fi} = \begin{bmatrix} A & B_2B_2' \\ 0 & -A' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a^2 & 0 & 0 & 1 \\ 0 & 0 & 0 & -a^2 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Its eigenvalues are $\pm a$ (double multiplicity each). Hence the stable invariant subspace of $\mathcal{H}_{fi}$ is the kernel of $(\mathcal{H}_{fi} + aI)^2$. A basis in this kernel is given by

$$\begin{bmatrix} 1 \\ 0 \\ -2a^3 \\ -2a^2 \end{bmatrix},\quad \begin{bmatrix} -1/a \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Hence

$$P_{fi} = -\begin{bmatrix} -2a^3 & 0 \\ -2a^2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1/a \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2a^3 & 2a^2 \\ 2a^2 & 2a \end{bmatrix},$$

and the optimal state feedback gain is given by

$$K = -B_2'P_{fi} = \begin{bmatrix} -2a^2 & -2a \end{bmatrix}.$$

A good sanity check here: the closed loop poles (eigenvalues of $A + B_2K$) should be identical to the stable eigenvalues of the Hamiltonian.

The state estimation Hamiltonian is

$$\mathcal{H}_{se} = \begin{bmatrix} A' & C_2'C_2 \\ B_1B_1' & -A \end{bmatrix} = \begin{bmatrix} 0 & a^2 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & -a^2 & 0 \end{bmatrix}.$$

Its characteristic polynomial is $s^4 - 2a^2s^2 + a^4 + 1$, and hence its eigenvalues s satisfy $s^2 = a^2 \pm j$. The stable eigenvalue $s = x + jy$ such that $s^2 = a^2 + j$ is given by

$$x = -\sqrt{\frac{\sqrt{a^4+1}+a^2}{2}},\ y = -\sqrt{\frac{\sqrt{a^4+1}-a^2}{2}}.$$

The corresponding eigenvector is

$$h = \begin{bmatrix} x+jy \\ 1 \\ j \\ y-jx \end{bmatrix}.$$

Since $\mathcal{H}_{se}$ has real coefficients, the eigenvector corresponding to eigenvalue $x - jy$ will be the complex conjugate of h. Hence real and imaginary parts of h form a basis in the stable invariant subspace of $\mathcal{H}_{se}$:

$$\begin{bmatrix} y \\ 0 \\ 1 \\ -x \end{bmatrix}, \quad \begin{bmatrix} x \\ 1 \\ 0 \\ y \end{bmatrix}.$$

Hence

$$P_{se} = -\begin{bmatrix} 1 & 0 \\ -x & y \end{bmatrix} \begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1/y & x/y \\ x/y & -x^2/y - y \end{bmatrix},$$

and

$$L = P_{se}C_2' = \begin{bmatrix} 1/y \\ -x/y \end{bmatrix}.$$

(c) Derive an analytical expression for the transfer function of the optimal dynamic feedback controller, and verify it using numerical calculations with `h2syn.m`.

The closed loop transfer function is given by

$$G(s) = -K(sI - A - B_2K - LC_2)^{-1}L.$$

To verify the formulae, MATLAB function `ps3_1.m` can be used. This function relies on SIMULINK design diagram `ps3_1a.m`.

Problem 3.2

Random signal $q = q(t)$ is assumed to be a "bandlimited white noise" of a given bandwidth B (i.e. the result of passing the true white noise $v_1(t)$ through an ideal low-pass filter of bandwidth B rad/sec). A high quality sensor is assumed to measure $q(t)$ accurately, except for a white additive noise, with the signal-to-noise ratio of 10.

(a) Use `h2syn.m` to design a 10-th order linear filter which inputs the sensor output, and outputs an estimate of $\dot{q}(t)$ which makes the mean square estimation error as small as possible.

M-function `ps3_2a.m` does the job. It uses a 10-th order Butterworth filter W with cut-off frequency B to model q as $q = Ww$, where w is the normalized white noise.

(b) Test your design by comparing the *simulated* performance of filters you have designed for $B = 10$ rad/sec and $B = 1$ rad/sec on signals $q(\cdot)$ of bandwidths of $B = 10$ and $B = 1$ rad/sec. (One expects that the filter optimized for $B = 10$ rad/sec will be better on the $q(\cdot)$ with bandwidth $B = 10$ rad/sec than the filter optimized for $B = 1$ rad/sec, and vice versa.) Use the generator of bandlimited white noise supplied with the SIMULINK to perform the simulations.

The SIMULINK diagram for testing is `ps3_2c.mdl`. You must run `ps3_2b.m` before you open it. For $B = 1$ rad/sec, the degradation of performance when a filter designed for $B = 10$ is used is dramatic (at least a 10-fold increase of error). For $B = 10$, the degradation of performance when a filter designed for $B = 1$ is used is not as big, but still quite noticeable.