Massachusetts Institute of Technology

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6.245: MULTIVARIABLE CONTROL SYSTEMS

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Problem Set 2 Solutions [1]

Problem 2.1

FOR EACH OF THE STATEMENTS BELOW DECIDE WHETHER IT IS TRUE OR FALSE. FOR A TRUE STATEMENT, SKETCH A PROOF. FOR A FALSE STATEMENT, GIVE A COUNTEREXAMPLE.

(a) H2 NORM OF A STABLE FINITE ORDER CT LTI STATE SPACE MODEL IS NEVER LARGER THAN 100 TIMES ITS H-INFINITY NORM.

This statement is **false**. To see this, consider the stable first order LTI CT system with transfer function

$$G(s) = G_a(s) = \frac{1}{s+a},$$

where $a > 0$ is a real parameter. Then

$$\|G_a\|_\infty = \sup_{\omega \in \mathbf{R}} |G(j\omega)| = \sup_{\omega \in \mathbf{R}} \frac{1}{\sqrt{a^2+\omega^2}} = \frac{1}{a},$$

$$\|G_a\|_{H2} = \left(\int_0^\infty e^{-2at} dt \right)^{1/2} = \frac{1}{\sqrt{2a}}$$

(we used the fact that $g_a(t) = e^{-at}$, where $t \geq 0$, is the impulse response of G_a). Hence, as $a \to 0$, H-Infinity norm of G_a becomes arbitrarily large relative the H2 norm of G_a.

[1] Version of February 22, 2004

(b) H-Infinity norm of a stable finite order CT LTI state space model is never larger than 100 times its H2 norm.

This statement is **false**. To see this, use G_a from (a) with $a \to \infty$.

(c) H2 norm of a stable finite order DT LTI state space model is never larger than 100 times its H-Infinity norm.

The statement is *true* for SISO models, because

$$\|G\|_{H2}^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |G(e^{j\omega})|^2 d\omega \le \frac{1}{2\pi}\int_{-\pi}^{\pi} \max_{\omega}\{|G(e^{j\omega})|^2\} d\omega = \|G\|_\infty^2.$$

However, for MIMO systems, the statement is *false*. To see this, consider $G = G(z)$ which is an n-by-n identity matrix: its H-Infinity norm equals 1 while the H2 norm equals $\sqrt{n}$.

(d) H-Infinity norm of a stable finite order DT LTI state space model is never larger than 100 times its H2 norm.

This statement is **false**. To see this, consider the stable first order LTI DT system with transfer function

$$H(z) = H_a(z) = \frac{1}{1 - a/z},$$

where $a \in (0, 1)$ is a real parameter. Then

$$\|H_a\|_\infty = \sup_{\omega \in \mathbf{R}} |H(e^{j\omega})| = \frac{1}{1-a},$$

$$\|H_a\|_{H2} = \left(\sum_{k=0}^{\infty} a^{2k}\right)^{1/2} = \frac{1}{\sqrt{1-a^2}}$$

(we used the fact that $h_a[k] = a^k$, where $k \ge 0$, is the impulse response of H_a). Hence, as $a \to 1$, H-Infinity norm of H_a becomes arbitrarily large relative the H2 norm of H_a.

Problem 2.2

For continuous time (non-LTI) systems S_a with scalar input $f = f(t)$ and scalar output $g = g(t)$, described below, find their L2 gains, as functions of parameter $a > 0$. Support your answer with arguments (typically,

TO SHOW THAT L2 GAIN OF A SYSTEM EQUALS, SAY, 0.5, ONE HAS TO FIND INPUT/OUTPUT PAIRS OF INFINITE ENERGY (I.E. NOT CONVERGING TO ZERO) FOR WHICH THE ASYMPTOTIC (AS TIME CONVERGES TO INFINITY) OUTPUT-TO-INPUT ENERGY RATIO IS ARBITRARILY CLOSE TO $0.25 = 0.5^2$, AND, IN ADDITION, TO SHOW THAT THE ASYMPTOTIC ENERGY RATIO CANNOT BE LARGER THAN 0.5).

(a) $g(t) = a\sin(f(t))$;

L2 gain equals a.

To see that the L2 gain cannot be larger, note that

$$|\sin(y)| \leq |y|$$

for all real y. Hence

$$\int_0^T |g(t)|^2 dt = \int_0^T |a\sin(f(t))|^2 dt \leq a^2 \int_0^T |f(t)|^2 dt.$$

To see that the L2 gain cannot be smaller, consider input $f(t) \equiv e$, where $\epsilon > 0$ is a small parameter. Then, for every γ we have

$$\int_0^T \{\gamma^2 |f(t)|^2 - |g(t)|^2\} dt = T(\gamma^2\epsilon^2 - a^2\sin^2(\epsilon)),$$

which will converge to minus infinity as $T \to \infty$ unless

$$\gamma^2\epsilon^2 - a^2\sin^2(\epsilon) \geq 0.$$

Since this inequality must be satisfied for all ϵ whenever γ is larger than the L2 gain, we have $\gamma^2 \geq a^2$.

(b) $g(t) = f(at)\sin(t)$;

For $a \leq 1$ L2 gain equals $1/\sqrt{a}$. For $a > 1$ the gain is infinite (and the system is not causal).

To see that L2 gain does not exceed $1/\sqrt{a}$ for $a \leq 1$, note that

$$\int_0^T |g(t)|^2 dt = \int_0^T |f(at)|^2 \sin^2(t) dt$$

$$\leq \int_0^T |f(at)|^2 dt = \frac{1}{a}\int_0^{aT} |f(t)|^2 dt \leq \frac{1}{a}\int_0^T |f(t)|^2 dt.$$

To see that L2 gain is not smaller than $1/\sqrt{a}$ for $a \le 1$, consider

$$f(t) = \begin{cases} k^{-1/2}, & t \in [(k+\frac{1}{2})\pi - \epsilon, (k+\frac{1}{2})\pi + \epsilon],\ k \in \{1, 2, \dots\}, \\ 0, & \text{otherwise.} \end{cases}$$

(The main idea is that $f(t)$ should be zero when $|\sin(t)|$ is not close to 1, and the energy of $f(t)$ on the interval $[aT, T]$ should converge to zero as $T \to \infty$, while the total energy of f should be infinite.) Then

$$\int_0^T \{\gamma^2 |f(t)|^2 - |g(t)|^2\} dt \le \int_0^T \gamma^2 |f(t)|^2 dt - \int_0^T \cos^2(\epsilon) |f(at)|^2 dt$$
$$= \left(\gamma^2 - \frac{\cos^2(\epsilon)}{a}\right) \int_0^{aT} |f(t)|^2 dt + \gamma^2 \int_{aT}^T |f(t)|^2 dt,$$

which converges to $-\infty$ as $T \to \infty$ unless $\gamma^2 \ge \cos^2(\epsilon)/a$. Since $\epsilon > 0$ can be arbitrarily small, $g \ge 1/a$ for $a \ge 1$.

To show that L2 gain is infinite for $a > 1$, consider

$$f(t) = \begin{cases} r^k, & t \in [(k+\frac{1}{2})\pi - \epsilon, (k+\frac{1}{2})\pi + \epsilon],\ k \in \{1, 2, \dots\}, \\ 0, & \text{otherwise,} \end{cases}$$

where $r \gg 1$ is a parameter. It is easy to see that, when $r \to \infty$ is sufficiently large compared to γ, the integrals

$$\int_0^T \{\gamma^2 |f(t)|^2 - |g(t)|^2\} dt$$

converge to minus infinity as $T \to \infty$.

(c) $g(t) = af(t) - |f(t-1)|$.

L2 gain equals $1 + a$.

To see that L2 gain does not exceed $1 + a$, note that

$$|ax + y|^2 \le (1+a)\left(a|x|^2 + |y|^2\right) \quad \forall\ x, y,$$

and hence

$$\int_0^T |g(t)|^2 dt \le (1+a)a \int_0^T |f(t)|^2 dt + (1+a) \int_0^T |f(t-1)|^2 dt$$
$$\le (1+a)^2 \int_0^T |f(t)|^2 dt + \int_{-1}^0 |f(t)|^2 dt.$$

To see that L2 gain is not smaller than $1 + a$, consider the case when $f(t) \equiv -1$, and hence $g(t) \equiv -(1+a)$.

Problem 2.3

Continuous time signal $q = q(t)$ is the output of a pure double integrator system with input $f_1 = f_1(t)$, and $g(t) = q(t) + bf_2(t)$, where $b > 0$ is a known constant (do the calculations for $b = 0.1$ and $b = 10$). Find an LTI filter $F = F(s)$ which takes $g = g(t)$ as an input and outputs an estimate $\hat{q} = \hat{q}(t)$ of $q = q(t)$, which is "good" in one of the following interpretations:

(a) Assuming that $f = [f_1; f_2]$ is white noise, minimize the asymptotic value of the variance of the estimation error $e = q - \hat{q}$.

(b) Minimize the L2 gain from $f = [f_1; f_2]$ to the estimation error $e = q - \hat{q}$ with accuracy 10 percent.

For a system with state space equations

$$\dot{x}(t) = A^0 x(t) + B_1^0 w(t)$$

and sensor output

$$y_0(t) = C_2^0 x(t) + D_{21}^0 w(t),$$

a standard observer has the format

$$\dot{\hat{x}}(t) = A^0 \hat{x}(t) + L(C_2^0 \hat{x}(t) - y_0(t)),$$

where matrix L is chosen in such way that $A^0 + LC_2^0$ is a Hurwitz matrix. Note that here $\hat{x}$ is a result of applying an LTI transformation to y. Hence

$$d(t) = y_0(t) - C_2^0 \hat{x}(t)$$

is a result of applying an LTI transformation to $y_0(t)$, and, reciprocally, $y_0(t)$ is a result of applying an LTI transformation to $d(t)$. Therefore, designing an LTI filter F with input y_0 output $v = Fy_0$, to be an optimal estimate of a state component $q(t) = C_1^0 x(t)$ can be reduced to designing an LTI filter F_d with input $d(t)$ and output $v_d = F_d d$, to be a good estimate of $q_d(t) = q(t) - C_1^0 \hat{x}(t)$: the relation between v_d and v will be

$$v(t) = C_1^0 \hat{x}(t) + v_d(t).$$

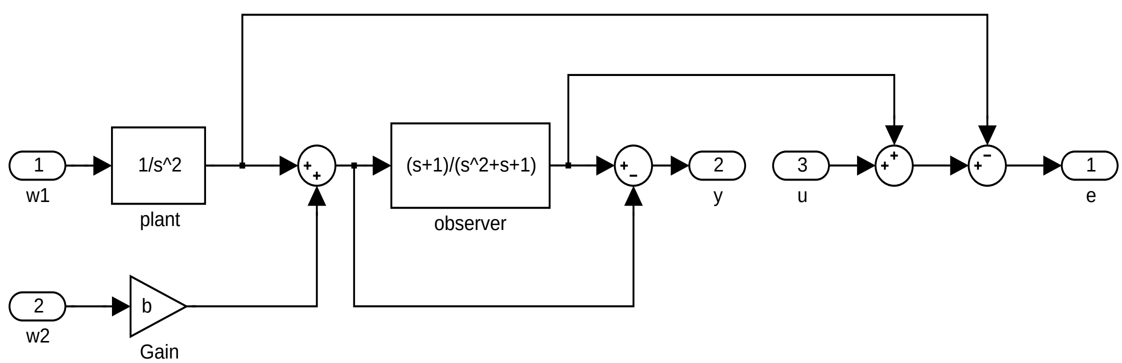
The relation between w, d, and q_d is given by

$$\dot{e} = (A^0 + LC_2^0)e + (B_1^0 + LD_{21}^0)w, \;\; d = C_2^0 e + D_{21}^0 w, \;\; q_d = C_1^0 e,$$

where $e(t) = x(t) - \hat{x}(t)$ plays the role of system state. The task of finding an optimal or suboptimal estimator for q_d based on measuring d can be formulated as a standard LTI feedback optimization setup with

$$A = A^0 + LC_2^0,\ B_1 = B_1^0 + LD_{21}^0,\ B_2 = 0,\ C_1 = C_1^0,\ D_{11} = 0,\ D_{12} = I,\ C_2 = C_2^0,\ D_{21} = D_{21}^0,\ D_{22} = 0.$$

To implement the filter optimization approach using MATLAB, we use a design SIMULINK model `ps2_3a.mdl`,



handled by M-function `ps2_3.mdl`:

```
function [Fh2,Fhi,Eh2,Ehi]=ps2_3(b)
% function ps2_3(b)
%
% solution for Problem 2.3 in 6.245/Spring 2004

if nargin<1, b=1; end
assignin('base','b',b);
s=tf('s');
assignin('base','s',s);
load_system('ps2_3a');
[a,b,c,d]=linmod('ps2_3a');
close_system('ps2_3a');
[ar,br,cr,dr]=ssdata(minreal(ss(a,b,c,d)));
p=pck(ar,br,cr,dr);
nmeas=1;
ncon=1;
```

```
ricmethd=2;
quiet=0;
[kh2,gh2]=h2syn(p,nmeas,ncon,ricmethd,quiet);
[ah2,bh2,ch2,dh2]=unpck(kh2);
[ag,bg,cg,dg]=unpck(gh2);
Kh2=ss(ah2,bh2,ch2,dh2);
G=(s+1)/(s^2+s+1);
disp('H2 controller:')
Fh2=tf(minreal(Kh2*(G-1)+G))
Eh2=tf(minreal(ss(ag,bg,cg,dg)));
gmin=0;
gmax=norm(Eh2,Inf);
tol=0.01;
epr=1e-10;
epp=1e-6;
[khinf,ghinf]=hinfsyn(p,nmeas,ncon,gmin,gmax,tol,ricmethd,epr,epp,quiet);
[ahi,bhi,chi,dhi]=unpck(khinf);
[ag,bg,cg,dg]=unpck(ghinf);
Ehi=tf(minreal(ss(ag,bg,cg,dg)));
Khi=ss(ahi,bhi,chi,dhi);
disp('H-Infinity controller:')
Fhi=tf(minreal(Khi*(G-1)+G))
```

Note the need for using `minreal.m`: MATLAB does not eliminate uncontrollable/unobservable states automatically.