Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.243j (Fall 2003): DYNAMICS OF NONLINEAR SYSTEMS

by A. Megretski

Take-Home Test 1 Solutions[1]

Problem T1.1

FIND ALL VALUES OF $\mu \in \mathbf{R}$ FOR WHICH THE FUNCTION $V:\ \mathbf{R}^2 \mapsto \mathbf{R}$, DEFINED BY

$$V\left(\left[\begin{array}{c}\bar{x}_1\\ \bar{x}_2\end{array}\right]\right) = \max\{|\bar{x}_1|, |\bar{x}_2|\}$$

IS MONOTONICALLY NON-INCREASING ALONG SOLUTIONS OF THE ODE

$$\left\{\begin{array}{rcl}\dot{x}_1(t) & = & \mu x_1(t) + \sin(x_2(t)),\\ \dot{x}_2(t) & = & \mu x_2(t) - \sin(x_1(t)).\end{array}\right.$$

Answer: $\mu \leq -1$.

Proof For $\mu \leq -1$, $x_1 \neq 0$ we have

$$\begin{array}{rcl}\frac{1}{2}\frac{d}{dt}x_1^2 & = & \mu x_1^2 + x_1\sin(x_2)\\ & < & -|x_1|(|x_1| - |x_2|),\end{array}$$

and hence $|x_1|$ is strictly monotonically decreasing when $x \neq 0$ and $|x_1| \geq |x_2|$. Similarly, $|x_2|$ is strictly monotonically decreasing when $x \neq 0$ and $|x_2| \geq |x_1|$. Hence, when $\mu \leq -1$, $V(x)$ is strictly monotonically decreasing along non-equilibrium trajectories of the system.

For $\mu > 1$, $x_1(0) = r$, $x_2(0) = r$, where $r > 0$ is sufficiently small we have

$$\dot{x}_1(0) = \mu r - \sin(r) > 0,$$

hence

$$V(x(t)) \geq x_1(t) > r = V(x(0))$$

when $t > 0$ is small enough, which proves that V is not monotonically decreasing.

■

[1] Version of October 20, 2003

Problem T1.2

FIND ALL VALUES OF $r \in \mathbf{R}$ FOR WHICH DIFFERENTIAL INCLUSION OF THE FORM

$$\dot{x}(t) \in \eta(x(t)), \quad x(0) = \bar{x}_0,$$

WHERE $\eta: \ \mathbf{R}^2 \mapsto 2^{\mathbf{R}^2}$ IS DEFINED BY

$$\eta(\bar{x}) = \{f(\bar{x}/|\bar{x}|)\} \quad \text{for} \quad \bar{x} \neq 0,$$

$$\eta(0) = \{f(y) : \ y = [y_1; y_2] \in \mathbf{R}^2, \ |y_1| + |y_2| \leq r\},$$

HAS A SOLUTION $x : \ [0, \infty) \mapsto \mathbf{R}^2$ FOR EVERY CONTINUOUS FUNCTION $f : \ \mathbf{R}^2 \mapsto \mathbf{R}^2$ AND FOR EVERY INITIAL CONDITION $\bar{x}_0 \in \mathbf{R}^2$.
Answer: $r \geq \sqrt{2}$.
Proof First, let us show that existence of solutions is not guaranteed when $r < \sqrt{2}$. Let $\epsilon > 0$ be such that $2\epsilon < \sqrt{2} - r$. Define

$$f\left(\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right]\right) = \left[\begin{array}{c} 0.5\sqrt{2}(1-\epsilon) - x_1 \\ 0.5\sqrt{2}(1-\epsilon) - x_2 \end{array}\right].$$

Let us show that, for this f, the differential inclusion $\dot{x}(t) =\in \eta(x(t))$ will have no solutions $x : \ [0, \infty) \mapsto \mathbf{R}^2$ with $x(0) = 0$. Indeed, since

$$x' f(x/|x|) < 0 \ \ \forall \ \ x \neq 0, x \in \mathbf{R}^2,$$

$|x(t)|$ is strictly monotonically decreasing when $x(t) \neq 0$. Therefore $x(0) = 0$ implies $x(t) = 0$ for $t \geq 0$. Hence $\dot{x}(t) = 0 \in \eta(0)$. However, for $r < \sqrt{2}$, and for this particular selection of $f(\cdot)$, zero is not an element of $\eta(0)$. The contradiction shows that no solution with $x(0) = 0$ exists.

To prove existence of solutions for $r \geq \sqrt{2}$, one is tempted to use the existence theorem relying on convexity and semicontinuity of $\eta(\cdot)$. However, these assumptions are not necessarily satisfied in this case, since the set $\eta(0)$ does not have to be convex. Instead, note that, by the continuity of f, existence of a solution $x : \ [t_0, t_0 + |\bar{x}_0|/M) \mapsto \mathbf{R}^2$ with $x(t_0) = \bar{x}_0$ is guaranteed for all $\bar{x}_0 \neq 0$. Hence, it is sufficient to show that a solution $x_0 : \ [0, \infty) \mapsto \mathbf{R}^2$ with $x_0(0) = 0$ exists.

To do this, consider two separate cases: $0 \in \eta(0)$ and $0 \notin \eta(0)$. If $0 \in \eta(0)$ then $x(t) \equiv 0$ is the desired solution of the differential inclusion. Let us show that $0 \notin \eta(0)$ implies existence of a solution $q \in (0, \infty)$, $|u| = 1$ of the equation $f(u) = qu$. Indeed, if $0 \notin \eta(0)$ and $r \geq \sqrt{2}$ then $0 \neq f(\tau u)$ for all $\tau \in [0, 1]$, $|u| = 1$, and hence

$$(\tau, u) \mapsto \frac{f(\tau u)}{|f(\tau u)|}$$

is a homotopy between the vector fields $f_1: u \mapsto f(u)/|f(u)|$ and $f_0: u \mapsto f(0)/|f(0)|$. Since the index of the *constant* map f_0 is zero, the index of f_1 is zero as well. However, assuming that $f(u) \neq qu$ for $q \in (0,\infty)$, $|u| = 1$ yields a homotopy

$$(\tau, u) \mapsto \frac{\tau u + (1-\tau)f(u)}{|\tau u + (1-\tau)f(u)|}$$

between f_1 and the identity map, which is impossible, since the identity map has index 1.

Hence $f(u) = qu$ for some $q > 0$, $|u| = 1$, which yields $x_0(t) = qtu$ as as a valid solution $x_0: [0,\infty) \mapsto \mathbf{R}^2$ of the differential inclusion.

■

Problem T1.3

FIND ALL VALUES $q, r \in \mathbf{R}$ FOR WHICH $\bar{x}_0 = 0$ IS *not* A (LOCALLY) STABLE EQUILIBRIUM OF THE ODE

$$\dot{x}(t) = Ax(t) + B(Cx(t))^{1/3} \tag{1.1}$$

FOR EVERY SET OF MATRICES A, B, C OF DIMENSIONS n-BY-n, n-BY-1, AND 1-BY-n RESPECTIVELY, SUCH THAT A IS A HURWITZ MATRIX AND

$$\mathrm{Re}[(1 + j\omega q)G(j\omega)] > r \quad \forall\, \omega \in \mathbf{R} \tag{1.2}$$

FOR

$$G(s) = C(sI - A)^{-1}B.$$

Answer: $r \geq 0$, $q \in \mathbf{R}$ arbitrary (note, however, that for $r \geq 0$ (1.2) implies $q \geq 0$).

Proof If $r < 0$, take $A = -1$, $B = 0$, $C = 1$ to get an example of A, B, C satisfying the conditions and such that $\bar{x}_0 = 0$ is a (globally asymptotically) stable equilibrium of (1.1).

Now consider the case $r \geq 0$. Then, informally speaking, the frequency domain condition means some sort of "passivity" of G, while (1.1) describes a positive feedback interconnection of G with nonlinearity $y \mapsto w = y^{1/3}$, which can be characterized as having arbitrarily large positive gain for $x \approx 0$. Hence one expects instability of the zero equilibrium of (1.1).

To show local instability, let us prove existence of a Lyapunov function $V = V(x)$ for which 0 is not a local minimum, and

$$\frac{d}{dt}V(x(t)) < 0 \quad \text{whenever} \quad |Cx(t)| \in (0, \epsilon_0)$$

for some $\epsilon_0 > 0$. Note that this will imply instability of the equilibrium $\bar{x}_0 = 0$, since every solution with $V(x(0)) < V(0)$ and $|x(0)| < \epsilon_0/2|C|$ will eventually cross the sphere $|x(0)| = \epsilon_0/2|C|$ (otherwise $|Cx(t)| \leq \epsilon_0/2$ for all $t \geq 0$, hence $V(x(t))$ is monotonically non-increasing, and all limit points $\bar{x}$ of $x(\cdot)$ satisfy $C\bar{x} = 0$, therefore every solution $x_*(t)$ of (1.1) beginning at such limit point satisfies $Cx_*(t) = 0$ and hence converges to the origin, which contradicts $V(x(0)) < V(0)$.

By introducing $w(t) = (Cx(t))^{1/3}$, system equations can be re-written in the form

$$\dot{x}(t) = Ax(t) + Bw(t).$$

Consider first the (simpler) case when $r > 0$ (and hence $q > 0$). Then one can use the inequality

$$w(t)Cx(t) \leq r|w(t)|^2,$$

for sufficiently small $|Cx(t)|$. Condition (1.2) together with the KYP Lemma yields existence of a matrix $P = P'$ such that

$$\bar{w}C\bar{x} + q\bar{w}C(A\bar{x} + B\bar{w}) - r|\bar{w}|^2 \geq 2\bar{x}'P(A\bar{x} + B\bar{w}) \ \ \forall\ \bar{x} \in \mathbf{R}^n, \bar{w} \in \mathbf{R}.$$

Substituting $w = (Cx)^{1/3}$, we get

$$\frac{d}{dt}[x'Px - 0.75q|Cx|^{4/3}] \leq |Cx|^{4/3} - r|Cx|^{2/3},$$

which is exactly what is needed, because $y^{4/3} - ry^{2/3} < 0$ for $y \in (0, \sqrt{r})$. In addition, for every $\bar{x}_0 \in \mathbf{R}^n$ such that $C\bar{x}_0 \neq 0$ the expression

$$V(\bar{x}) = \bar{x}'P\bar{x} - 0.75q|C\bar{x}|^{4/3}$$

is negative when $\bar{x} = r\bar{x}_0$ and $r > 0$ is small enough.

To prove the answer in the general case, note that the inequality

$$w(t)Cx(t) > R|Cx(t)|^2$$

is satisfied whenever $|Cx(t)| \in (0, \epsilon)$ with $\epsilon > 0$ and $R = \epsilon^{-2/3}$, i.e. R can be made arbitrarily large by selecting an appropriate $\epsilon > 0$. Therefore the derivative bound for $V(x(t)) = x(t)'Px(t)$ will hold if

$$2\bar{x}'P(A\bar{x} + B\bar{w}) \leq R|C\bar{x}|^2 - \bar{w}C\bar{x} \ \ \forall\ \bar{x} \in \mathbf{R}^n, \bar{w} \in \mathbf{R}. \tag{1.3}$$

According to the KYP Lemma, such $P = P'$ exists if

$$R|G(j\omega)|^2 - \mathrm{Re}(G(j\omega)) > 0 \ \ \forall\ \omega \in \mathbf{R},$$

or, equivalently,

$$\mathrm{Re}(1/G(j\omega)) < R \ \ \forall\ \omega \in \mathbf{R}.$$

Moreover, substituting $w = (R + K)Cx$, where $K > 0$ is a constant, into (1.3) yields

$$P(A + BKC) + (A + BKC)'P \leq -KC'C.$$

Therefore, $P = P'$ cannot be positive semidefinite if $A + BKC$ has eigenvalues with positive real part.

We will rely on the following statement from the linear system theory: if $H(s)$ is a stable proper rational transfer function which is positive real (i.e. $\mathrm{Re}(H(j\omega)) > 0$ for all $\omega \in \mathbf{R}$) then $\mathrm{Re}(s) > 0$ whenever $\mathrm{Re}(s) > 0$, and the relative degree of H is not larger than one.

Consider $H(s) = (1 + qs)G(s)$. By assumption, H is positive real and proper. Hence $q \geq 0$ (otherwise $H(-1/q) = 0$). If relative degree of H is zero then $q > 0$, and hence $sG(s)$ converges to a non-zero limit H_* as $s \to \infty$. Since $(1 + qr)G(r) > 0$ for $r > 0$, it follows that $H_* > 0$, and hence

$$\mathrm{Re}\frac{1}{G(j\omega)} = \mathrm{Re}\frac{j\omega}{G(j\omega)}$$

is bounded as $\omega \to \infty$.

If relative degree of H is one then $sH(s)$ converges to a positive limit as $s \to \infty$, and hence

$$\mathrm{Re}\frac{1}{G(j\omega)} = \mathrm{Re}\frac{j\omega - (j\omega)^2}{(j\omega)^2 G(j\omega)}$$

is bounded from above as $\omega \to \infty$. Finally, since $G(r) > 0$ and $G(r) \to 0$ as $r \to +\infty$, it follows that the equation $1 = KG(r)$ has a positive solution r for all sufficiently large $K > 0$. Hence matrix $A + BKC$ has a positive eigenvalue for all sufficiently large $K > 0$.

$\blacksquare$