Massachusetts Institute of Technology

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6.243j (Fall 2003): DYNAMICS OF NONLINEAR SYSTEMS

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Problem Set 7 Solutions[1]

Problem 7.1

A stable linear system with a relay feedback excitation is modeled by

$$\dot{x}(t) = Ax(t) + B\mathrm{sgn}(Cx(t)), \tag{7.1}$$

where A is a Hurwitz matrix, B is a column matrix, C is a row matrix, and $\mathrm{sgn}(y)$ denotes the sign nonlinearity

$$\mathrm{sgn}(y) = \begin{cases} 1, & y > 0, \\ 0, & y = 0, \\ -1, & y < 0. \end{cases}$$

For $T > 0$, a $2T$-periodic solution $x = x(t)$ of (7.1) is called a *regular unimodal limit cycle* if $Cx(t) = -Cx(t+T) > 0$ for all $t \in (0, T)$, and $CAx(0) > |CB|$.

(a) Derive a necessary and sufficient condition of exponential local stability of the regular unimodal limit cycle (assuming it exists and A, B, C, T are given).

Let Y denote the set of all $\bar{x} \in \mathbf{R}^n$ such that $C\bar{x} = 0$.

Let $x_0 = x(0)$. By assumptions, $Cx(t) > 0$ and $Cx(-t) = Cx(T - t + T) = -Cx(T - t) < 0$ for $t \in (0, T)$. Hence $Cx(0) = Cx_0 = 0$, i.e. $x_0 \in Y$.

Let $F :\ \mathbf{R} \times Y$ be defined by

$$F(t, \bar{x}) = e^{At}(\bar{x} + A^{-1}B) - A^{-1}B.$$

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By definition $F(\tau, \bar{x})$ is the value at $t = \tau$ of the solution $z = z(t)$ of the ODE $dz/dt = Az + B$. Since $F(t, x_0) > 0$ for $t \in (0, T)$ and

$$\frac{dF}{dt}(0, \bar{x}) = C(A\bar{x} + B) \approx C(Ax_0 + B) > 0$$

whenever $\bar{x} \in Y$ is sufficiently close to x_0, we conclude that $F(t, \bar{x}) > 0$ for all $t \in (0, T)$ and for all $\bar{x} \in Y$ sufficiently close to x_0.

On the other hand,

$$\frac{dCF}{dt}(T, x_0) = C(Ax(T) + B) = -CAx_0 + CB < 0.$$

Hence, by the implicit mapping theorem, for $\bar{x} \in Y$ sufficiently close to x_0 equation $CF(t, \bar{x}) = 0$ has a unique solution $\bar{t} = h(\bar{x})$ in a neigborhood of $t = T$.

Consider the map S defined for $x_1 \in Y$ in a neigborhood of x_0 by $S(x_1) = F(h(x_1), x_1)$. Essentially, S is the Poincare map associated with the periodic solution $x = x(t)$. Local exponential stability of the trajectory of $x = x(t)$ is therefore equivalent to local exponential stability of the equilibrium x_0 of S.

The differential of S at x_0 is the composition of e^{AT} and the projection on Y parallel to $Ax(T) + B = B - Ax_0$. In other words, the differential of S has matrix

$$\dot{S}(x_0) = e^{AT} - [C(B - Ax_0)]^{-1}(B - Ax_0)Ce^{AT}$$

in the standard basis of $\mathbf{R}^n$. In order for the limit cycle $x = x(t)$ to be locally exponentially stable, all eigenvalues of this matrix should have absolute value smaller than 1.

(b) Use the result from (a) to find an example of system (7.1) with a Hurwitz matrix A and an *unstable* regular unimodal limit cycle.

The MATLAB code is provided in file `hw7_1_6243_2003.m`. To generate examples of unimodal lmit cycles, take a Hurwitz polynomial p and first constract A, B from a state space realization of transfer function $G(s) = 1/p(s)$. Use $T = 1$, and find x_0 from equation $F(T, x_0) = -x_0$, i.e.

$$x_0 = (I + e^{AT})^{-1}(I - e^{AT})A^{-1}B.$$

Then construct C such that $Cx_0 = 0$, $CB = 1$, and $CAx_0 = r$ where $r > 1$ is a parameter to be tuned up to achieve instability of the limit cycle. Check whether the resulting trajectory $x = x(t)$ is indeed a unimodal limit cycle by verifying the inequality $Cx(t) > 0$ for $t \in (0, T)$ (this step is not necessary when $n = 3$).

Numerical calculations show that using $n = 3$ and $r \approx 1$ typically yields an unstable unimodal limit cycle as, for example, with

$$p(s) = (s + 1)^3, \quad r = 1.5.$$

Problem 7.2

A linear system controlled by modulation of its coefficients is modeled by

$$\dot{x}(t) = (A + Bu(t))x(t), \tag{7.2}$$

where A, B are fixed n-by-n matrices, and $u(t) \in \mathbf{R}$ is a scalar control.

(a) Is it possible for the system to be controllable over the set of all non-zero vectors $\bar{x} \in \mathbf{R}^n$, $\bar{x} \neq 0$, when $n \geq 3$? In other words, is it possible to find matrices A, B with $n > 2$ such that for every non-zero $\bar{x}_0, \bar{x}_1$ there exist $T > 0$ and a bounded function $u : [0, T] \mapsto \mathbf{R}$ such that the solution of (7.2) with $x(0) = \bar{x}_0$ satisfies $x(T) = \bar{x}_1$?

The answer to this question is positive (examples exist for all $n > 1$). One such example is given by

$$A = 0.5(\alpha + \beta), \quad B = I + 0.5(\alpha - \beta),$$

where

$$\alpha = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

To show that the resulting system (7.2) is controllable over the set of non-zero states, note first that the auxiliary driftless system with three scalar controls

$$\dot{x} = \alpha x u_1 + \beta x u_2 + x u_3$$

satisfies the conditions of complete controllability for all $x \neq 0$. Indeed, the Lie bracket $g = [g_1, g_2]$ of the "linear" vector fields $g_k(x) = A_k x$ is given by $g(x) = Ax$, where $A = [A_1, A_2] = A_1 A_2 - A_2 A_1$is the commutant of matrices A_1 and A_2. Hence for $g_1(x) = \alpha x$, $g_2(x) = \beta x$, and $g_3 = [g_1, g_2]$ we have $g_3(x) = \gamma x$, where

$$\gamma = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Since the matrix

$$[x \;\; \alpha x \;\; \beta x \;\; \gamma x] = \begin{bmatrix} x_1 & x_2 & x_1 & -x_3 \\ x_2 & -x_1 & x_3 & x_2 \\ x_3 & x_3 & -x_2 & x_1 \end{bmatrix}$$

has full rank whenever $x = [x_1; x_2; x_3] \neq 0$, the auxiliary system is fully controllable for $x \neq 0$.

Since the auxiliary system is fully controllable for $x \neq 0$, it is also fully controllable using piecewise constant controls along the vector fields x, αx, βx. Note that the flow along αx is given by $S_\alpha^t(x) = e^{\alpha t}x$. Since $e^{2\pi\alpha} = I$, negative time flows along αx can be implemented using positive time flows. Same conclusion is also true for β. Since the flows along $(A+B)x = \alpha x + x$ and $(A-B)x = \beta x - x$ differ from the flows along αx and βx only in scaling of the trajectory, we conclude that for every non-zero $x_1, x_2 \in \mathbf{R}^3$ there exists a (picewise constant) control u which moves x_1 to ρx_2 for some $\rho > 0$. Therefore, for every non-zero $x_1, x_2 \in \mathbf{R}^3$ there exists a (picewise constant) control u which moves x_1 first to $\rho_\alpha x_\alpha$, then to $\rho_\beta x_\beta$, and then to ρx_2, where

$$x_\alpha = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad x_\beta = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Note that the line

$$\{cx_\alpha : \quad c \in \mathbf{R}\}$$

is invariant for flow defined by te vector field $\alpha x + x$, and the flow moves points of this line monotonically from the origin. Similarly, the line

$$\{cx_\beta : \quad c \in \mathbf{R}\}$$

is invariant for flow defined by te vector field $\beta x - x$, and the flow moves points of this line monotonically to the origin. Hence, there also exists a piecewise constant control u which moves x_1 first to $\rho_\alpha x_\alpha$, then to $c_\alpha \rho_\alpha x_\alpha$, then to $c_\alpha \rho_\beta x_\beta$, then to $c_\beta c_\alpha \rho_\beta x_\beta$ and then to $c_\alpha c_\beta \rho x_2$, where c_α, c_β are arbitrary positive numbers such that $c_\alpha \geq 1$ and $c_\beta \leq 1$. Selecting c_α, c_β in such a way that $c_\alpha c_\beta \rho = 1$ yields a trajectory from x_1 to x_2.

While the "theoretical" derivation above is easy to generalize to higher dimensions, there exists a rather simple explicit algorithm for moving from a given vector $x_1 \neq 0$ to a given vector $x_2 \neq 0$ using not more than five switches of the piecewise constant control value $u(t) \in \{-1, 1\}$.

(b) Is it possible for the system to be full state feedback linearizable in a neigborhood of some point $\bar{x}_0 \in \mathbf{R}^n$ for some $n > 2$?

The answer to this question is positive (examples exist for all $n \geq 1$).

To find an example, search for a *linear* output $y = Cx$ of relative degree n. This requires

$$CB \equiv 0, \; CAB \equiv 0, \; \dots \; CA^{n-2}B = 0, \; CA^{n-1}B\bar{x}_0 \neq 0.$$

In particular, for $n = 3$ one can take

$$C = [1 \; 0 \; 0], \;\; B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \;\; A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \;\; \bar{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Problem 7.3

A NONLINEAR ODE CONTROL MODEL WITH CONTROL INPUT u AND CONTROLLED OUTPUT y IS DEFINED BY EQUATIONS

$$\begin{aligned} \dot{x}_1 &= x_2 + x_3^2, \\ \dot{x}_2 &= (1 - 2x_3)u + a\sin(x_1) - x_2 + x_3 - x_3^2, \\ \dot{x}_3 &= u, \\ y &= x_1, \end{aligned}$$

WHERE a IS A REAL PARAMETER.

(a) OUTPUT FEEDBACK LINEARIZE THE SYSTEM OVER A LARGEST SUBSET X_0 OF $\mathbf{R}^3$.

For the new state vector $z = [z_1; z_2; z_3]$ let $z_1 = y = x_1$. Since dz_1/dt does not depend on u, let $z_2 = dz_1/dt = x_2 + x_3^2$. Since

$$\dot{z}_2 = u + a\sin(x_1) - x_2 + x_3 - x_3^2,$$

the relative degree of y equals two at all points $x \in \mathbf{R}^3$, and the modified conrol should be defined by

$$v = u + a\sin(x_1) - x_2 + x_3 - x_3^2.$$

To define z_3, search for a scalar function of x_1, x_2, x_3 for which the gradient is not parallel to $[1\ 0\ 0]$ and is orthogonal to vector $[0; 1 - 2x_3; 1]$. One such function is

$$z_3 = x_2 - x_3 + x_3^2.$$

The system equations in terms of z_1, z_2, z_3, v are linear:

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= v, \\ \dot{z}_3 &= a\sin(z_1) - z_3. \end{aligned}$$

(b) DESIGN A (DYNAMICAL) FEEDBACK CONTROLLER WITH INPUTS $x(t), r(t)$, WHERE $r = r(t)$ IS THE REFERENCE INPUT, SUCH THAT FOR EVERY BOUNDED $r = r(t)$ THE SYSTEM STATE $x(t)$ STAYS BOUNDED AS $t \to \infty$, AND $y(t) \to r(t)$ AS $t \to \infty$ WHENEVER $r = r(t)$ IS CONSTANT.

One such controller is given by

$$u = -k_p(x_1 - r) - k_d(x_2 + x_3^2) - a\sin(x_1) + x_2 - x_3 + x_3^2,$$

where k_p and k_d are arbitrary positive constants, which is equivalent to

$$v = -k_p(z_1 - r) - k_d z_2.$$

Since the corresponding equations for z_1, z_2 are those of a stable LTI system, z_1, z_2 remain bounded whenever r is bounded, and $z_1 \to r$ when r is constant. Since $dz_3/dt + z_3 = a\sin(z_1)$ is also bounded, z_3 remains bounded as well. Since the transformation from z back to x, given by

$$x_1 = z_1, \quad x_2 = z_2 - (z_2 - z_3)^2, \quad x_3 = z_2 - z_3,$$

is continuous, x is also bounded whenever r is bounded.

(c) FIND ALL VALUES OF $a \in \mathbf{R}$ FOR WHICH THE OPEN LOOP SYSTEM IS FULL STATE FEEDBACK LINEARIZABLE.

It is convenient to check the full state feedback linearizability conditions in n terms of the z state variable. Then

$$f\left(\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}\right) = \begin{bmatrix} z_2 \\ 0 \\ a\sin(z_1) - z_3 \end{bmatrix}, \quad \dot{f} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ a\cos(z_1) & 0 & -1 \end{bmatrix},$$

and hence

$$[f, g] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [f, [f, g]] = \begin{bmatrix} 0 \\ 0 \\ a\cos(z_1) \end{bmatrix}.$$

This means that the system is locally full state feedback linearizable (to a *controllable* system) whenever $a\cos(z_1) \neq 0$. For $a = 0$ the system *is* an uncontrollable LTI system. For $a \neq 0$ and $z_1 \neq 0$ the new coordinates

$$p_1 = z_3, \quad p_2 = a\sin(z_1) - z_3, \quad p_3 = a\cos(z_1)z_2 - a\sin(z_1) + z_3$$

and the new control variable

$$w = a\cos(z_1)v - a\sin(z_1)z_2^2 - a\cos(z_1)z_2 + a\sin(z_1) - z_3$$

linearize completely system equations.

(d) TRY TO DESIGN A DYNAMICAL FEEDBACK CONTROLLER WITH INPUTS $y(t), r(t)$ WHICH ACHIEVES THE OBJECTIVES FROM (B). TEST YOUR DESIGN BY A COMPUTER SIMULATION.

Since all nonlinear elements of the z equations are functions of the observable variable $y = z_1$, it is easy to construct a stable observer for the system:

$$\begin{aligned} \dot{\hat{z}}_1 &= \hat{z}_2 + k_1(y - \hat{z}_1), \\ \dot{\hat{z}}_2 &= u + a\sin(y) - \hat{z}_3 + k_2(y - \hat{z}_1), \\ \dot{\hat{z}}_3 &= a\sin(y) - \hat{z}_3, \end{aligned}$$

where k_1, k_2 are arbitrary positive coefficients. With this observer, the control action can be defined by

$$u = -k_p(\hat{z}_1 - r) - k_d\hat{z}_2 - a\sin(\hat{z}_1) + \hat{z}_3.$$