Massachusetts Institute of Technology

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6.243j (Fall 2003): DYNAMICS OF NONLINEAR SYSTEMS

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Problem Set 3 Solutions[1]

Problem 3.1

FIND OUT WHICH OF THE FUNCTIONS $V : \mathbf{R}^2 \to \mathbf{R}$,

(a) $V(x_1, x_2) = x_1^2 + x_2^2$;

(b) $V(x_1, x_2) = |x_1| + |x_2|$;

(c) $V(x_1, x_2) = \max |x_1|, |x_2|$;

ARE VALID LYAPUNOV FUNCTIONS FOR THE SYSTEMS

(1) $\dot{x}_1 = -x_1 + (x_1 + x_2)^3$, $\dot{x}_2 = -x_2 - (x_1 + x_2)^3$;

(2) $\dot{x}_1 = -x_2 - x_1(x_1^2 + x_2^2)$, $\dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2)$;

(3) $\dot{x}_1 = x_2|x_1|$, $\dot{x}_2 = -x_1|x_2|$.

The answer is: (b) is a Lyapunov function for system (3) - and no other valid pairs System/Lyapunov function in the list. Please note that, when we say that a Lyapunov function V is *defined* on a set U, then we expect that $V(x(t))$ should non-increase along all system trajectories in U. In the formulation of Problem 3.1, V is said to be defined on the whole phase space $\mathbf{R}^2$. Therefore, $V(x(t))$ must be non-increasing along *all* system trajectories, in order for V to be a valid Lyapunov function.

To show that (b) is a valid Lyapunov function for (3), note first that system (3) is defined by an ODE with a Lipschitz right side, and hence has the uniqueness of solutions property. Now, every point $(x_1, x_2) \in \mathbf{R}^2$ with $x_1 = 0$ or $x_2 = 0$ is an equilibrium of (3). Hence V is automatically valid at those points. At every other point in $\mathbf{R}^2$, V is

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differentiable, with $dV/dx = [\text{sgn}(x_1); \text{sgn}(x_2)]$ being the derivative. Hence $\nabla V(x)f(x) = x_1x_2 - x_1x_2 = 0$ at every such point, which proves that $V(x(t))$ is non-increasing (and non-decreasing either) along all non-equilibrium trajectories.

Below we list the "reasons" why no other pair yields a valid Lyapunov function. Of course, there are many other ways to show that.

For system (1) at $x = (2, 0)$, we have $\dot{x}_1 > 0$, $\dot{x}_2 < 0$, hence both $|x_1|$ and $|x_2|$ are increasing along system trajectories in a neigborhood of $x = (2, 0)$. Since all Lyapunov function candidates (a)-(c) increase when *both* $|x_1|$ and $|x_2|$ increase, (a)-(c) are not valid Lyapunov functions for system (1).

For system (2) at $x = (0.5, -0.5)$, we have $\dot{x}_1 > 0$, $\dot{x}_2 < 0$, hence both $|x_1|$ and $|x_2|$ increase along system trajectories in a neigborhood of $x = (0.5, -0.5)$.

For system (3) at $x = (2, 1)$, we have $\dot{x} = (2, -2)$, hence both $x_1^2 + x_2^2$ and $\max(x_1, x_2)$ are increasing along system trajectories in a neigborhood of $x = (2, 1)$.

Problem 3.2

Show that the following statement is not true. Formulate and prove a correct version: *if $V : \mathbf{R}^n \mapsto \mathbf{R}$ is a continuously differentiable functional and $a : \mathbf{R}^n \mapsto \mathbf{R}^n$ is a continuous function such that*

$$\nabla V(\bar{x})a(\bar{x}) \le 0 \quad \forall\ \bar{x} : \ V(\bar{x}) = 1, \tag{3.1}$$

then $V(x(t)) \le 1$ for every solution $x : [0, \infty) \to \mathbf{R}^n$ of

$$\dot{x}(t) = a(x(t)) \tag{3.2}$$

with $V(x(0)) \le 1$.

There are *two* important reasons why the statement is not true: first, $\nabla V(\bar{x})$ should be non-zero for all $\bar{x}$ such that $V(\bar{x}) = 1$; second, solution of $\dot{x} = a(x)$ with initial condition $x(0) = \bar{x}_0$ such that $V(\bar{x}_0) = 1$ should be unique. Simple counterexamples based on these considerations are given by

$$V(x) = x^2 + 1, \ a(\bar{x}) = 1, \ x(t) = t,$$

and

$$V(x) = x + 1, \ a(\bar{x}) = 1.5\bar{x}^{1/3}, \ x(t) = t^{1.5}.$$

One correct way to fix the problem is by requiring a strict inequality in (3.1). Here is a less obvious correction.

Theorem 3.1 *Let $V : \mathbf{R}^n \to \mathbf{R}$ be a continuously differentiable functional such that $\nabla V(\bar{x}) \neq 0$ for all $\bar{x}$ satisfying $V(\bar{x}) = 1$, and let $a : \mathbf{R}^n \to \mathbf{R}^n$ be a locally Lipschitz function such that condition (3.1) holds. Then $V(x(t)) \le 1$ for every solution $x : [t_0, t_\infty) \to \mathbf{R}^n$ of (3.2) with $V(x(0)) \le 1$.*

Proof It is sufficient to prove that for every $\bar{x}_0 \in \mathbf{R}^n$ satisfying the condition $V(\bar{x}_0) = 1$ there exists $d > 0$ such that $V(x(t)) \le 1$ for $0 \le t \le d$ for the solution $x(t)$ of (3.2) with $x(0) = \bar{x}_0$. Indeed, for $\epsilon \in (0,1)$ define x^ϵ as a solution of equation

$$\dot{x}(t) = -\epsilon \nabla V(x(t))' + a(x(t)), \ x(0) = \bar{x}_0. \tag{3.3}$$

By the existence theorem, solutions x^ϵ are defined on a non-empty interval $t \in [0, d]$ which does not depend on ϵ. Note that

$$dV(x^\epsilon(t))/dt = \nabla V(x^\epsilon(t))(-\epsilon \nabla V(x^\epsilon(t))' + a(x^\epsilon(t))) \le -\epsilon \|\nabla V(x^\epsilon(t))\|^2 < 0$$

whenever $V(x^\epsilon(t)) = 1$, and hence the same inequality holds whenever $x^\epsilon(t)$ is *close enough* to the set $\{x : V(x) = 1\}$. Hence $V(x^\epsilon(t)) \le 1$ for $t \in [0, d]$ for all ϵ. Now, continuous dependence on parameters implies that $x^\epsilon(t)$ converges for all $t \in [0, d]$ to $x(t)$. Hence

$$V(x(t)) = \lim_{\epsilon \to 0} V(x^\epsilon(t)) \le 1.$$

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Problem 3.3

THE OPTIMAL MINIMAL-TIME CONTROLLER FOR THE DOUBLE INTEGRATOR SYSTEM WITH BOUNDED CONTROL

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = u(t), \end{cases} \qquad |u(t)| \le 1$$

HAS THE FORM

$$u(t) = -\text{sgn}(x_1(t) + 0.5 x_2(t)^2 \text{sgn}(x_2(t))).$$

(a) FIND A LYAPUNOV FUNCTION $V : \mathbf{R}^2 \mapsto \mathbf{R}^2$ FOR THE CLOSED LOOP SYSTEM, SUCH THAT $V(x(t))$ IS STRICTLY DECREASING ALONG ALL SOLUTIONS OF SYSTEM EQUATIONS EXCEPT THE EQUILIBRIUM SOLUTION $x(t) \equiv 0$.

The original problem set contained a typo: a "-" sign in the expression for $u(t)$ was missing. For completeness, a solution which applies to this case is supplied in the next section.

A hint was given in the problem formulation, stressing that u is a *minimal time control*. What is important here is that it takes only finite time for for a system solution to reach the origin. Therefore, the amount of time it takes for the system to reach the origin can be used as a Lyapunov function. Let us verify this by inspection. System equations are Lipschitz continuous outside the curve

$$\Omega_0 = \{x = [x_1; x_2] : \ x_1 = -0.5 x_2 |x_2|\},$$

Solving them explicitly (outside Ω) yields

$$x(t) = \begin{bmatrix} c_1 + c_2 t - 0.5t^2 \\ c_2 - t \end{bmatrix} \quad \text{for} \quad x(t) \in \Omega_+ = \{x = [x_1; x_2] : \; x_1 > -0.5x_2|x_2|\},$$

$$x(t) = \begin{bmatrix} c_1 + c_2 t + 0.5t^2 \\ c_2 + t \end{bmatrix} \quad \text{for} \quad x(t) \in \Omega_- = \{x = [x_1; x_2] : \; x_1 < -0.5x_2|x_2|\}.$$

In addition, no solutions with initial condition $x(0) = [-0.5r^2; r]$ or $x(0) = [0.5r^2; -r]$, where $r > 0$, exists, unless the $\text{sgn}(\cdot)$ function is understood as the set-valued sign

$$\text{sgn}(y) = \begin{cases} \{1\}, & y > 0, \\ [-1, 1], & y = 0, \\ \{-1\}, & y < 0, \end{cases}$$

in which case the corresponding soltion trajectories lie in Ω_0. Finally, there is an equilibrium solution $x(t) \equiv 0$.

The corresponding Lyapunov function (time it take to reach the origin) is now easy to calculate, and is given by

$$V(x) = \begin{cases} x_2 + 2\sqrt{x_2^2/2 + x_1}, & \text{for } x_1 + x_2|x_2|/2 \geq 0, \\ -x_2 + 2\sqrt{x_2^2/2 - x_1}, & \text{for } x_1 + x_2|x_2|/2 \leq 0. \end{cases}$$

As expected, $dV/dt = -1$ along system trajectories, and $x = 0$ is the only global minimum of V.

(b) Find out whether the equilibrium remains asymptotically stable when the same controller is used for the perturbed system

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = -\epsilon x_1(t) + u(t), \end{cases} \qquad |u(t)| \leq 1,$$

where $\epsilon > 0$ is small.

The Lyapunov function $V(x)$ designed for the case $\epsilon = 0$ is not monotonically non-increasing along trajectories of the perturbed system ($\epsilon > 0$). Indeed, when

$$x_1 = -0.5r^2 + r^8, \quad x_2 = r > 0,$$

we have

$$\dot{V}(x(t)) = -\epsilon x_1 - 1 - \frac{x_1 x_2}{\sqrt{0.5x_2^2 + x_1}},$$

which is positive when $r > 0$ is small enough.

However, the stability can be established for the case $\epsilon > 0$ using an alternative Lyapunov function. One such function is

$$V_1(x) = \begin{cases} \epsilon^2 x_2^2 + (1 + \epsilon^4 |x_1|)^2, & \text{for } |x_1| \geq x_2^2/2, \\ \epsilon^2 x_2^2 + (1 + \epsilon^4 x_2^2/2)^2, & \text{for } |x_1| \leq x_2^2/2. \end{cases}$$

By considering the two regions $|x_1| \geq x_2^2/2$ and $|x_1| \leq x_2^2/2$ separately, it is easy to see that $dV_1(x(t))/dt \leq 0$, and $dV_1(x(t))/dt = 0$ only for

$$x(t) \in N = \{[x_1; x_2] : \ |x_1| \geq x_2^2/2\}.$$

Note that the origin is the only global minimum of V_1. Also, V_1 is continuous and all level sets of V_1 are bounded. Hence, if a solution of the system equations does not converge to the origin as $t \to \infty$, it must have a limit point $\bar{x}_* \neq 0$ such that, for the solution $x_*(t)$ of the system equations with $x_*(0) = \bar{x}_*$,

$$V(x_*(t)) = V(\bar{x}_*) > \min_{\bar{x} \in \mathbf{R}^2} V(\bar{x}) \quad \forall\, t \geq 0.$$

This implies that $x_*(t) \in N$ for all $t \geq 0$. However, no solution except the equilibrium can remain forever in N. Hence the equilibrium $x = 0$ is globally asymptotically stable.

Using the fact that a non-equilibrium solution of system equations cannot stay forever in the region where $\dot{V}(x(t)) = 0$, in order to prove stability of the equilibrium as demonstrated above, is referred to as the *La Salle's invariance principle.* Essentially, the formulation and a proof of this popular general result are contained in the solution above.

Problem 3.3 with typo

The optimal minimal-time controller for the double integrator system with bounded control

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = u(t), \end{cases} \qquad |u(t)| \leq 1$$

has the form

$$u(t) = \text{sgn}(x_1(t) + 0.5 x_2(t)^2 \text{sgn}(x_2(t))).$$

(a) Find a Lyapunov function $V : \mathbf{R}^2 \mapsto \mathbf{R}^2$ for the closed loop system, such that $V(x(t))$ is strictly decreasing along all solutions of system equations except the equilibrium solution $x(t) \equiv 0$.

The system is unstable (all solutions except $x(t) \equiv 0$ converge to infinity). However, this does not affect existence of strictly decreasing Lyapunov functions. For example,

$$V([x_1; x_2]) = \begin{cases} -x_2, & x_1 + 0.5x_2|x_2| > 0, \\ -x_2, & x_1 + 0.5x_2|x_2| = 0, \ x_2 \geq 0, \\ x_2, & x_1 + 0.5x_2|x_2| < 0, \\ x_2, & x_1 + 0.5x_2|x_2| = 0, \ x_2 \leq 0. \end{cases}$$

To show that V is valid, note that the trajectories of this system are given by

$$x(t) = \begin{bmatrix} c_1 + c_2 t + 0.5t^2 \\ c_2 + t \end{bmatrix}$$

when $x_1 + 0.5x_2|x_2| > 0$ or $x_1 + 0.5x_2|x_2| = 0$ and $x_2 \geq 0$, and by

$$x(t) = \begin{bmatrix} c_1 + c_2 t - 0.5t^2 \\ c_2 - t \end{bmatrix}$$

when $x_1 + 0.5x_2|x_2| < 0$ or $x_1 + 0.5x_2|x_2| = 0$ and $x_2 \leq 0$.

(b) FIND OUT WHETHER THE EQUILIBRIUM REMAINS ASYMPTOTICALLY STABLE WHEN THE SAME CONTROLLER IS USED FOR THE PERTURBED SYSTEM

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = -\epsilon x_1(t) + u(t), \end{cases} \qquad |u(t)| \leq 1,$$

WHERE $\epsilon > 0$ IS SMALL.

As can be expected, the equilibrium of the perturbed system is unstable just as the equilibrium of the unperturbed one is. To show this, note that for

$$x \in K = \{[x_1; x_2] : \ x_1 \in (0, 1/(2\epsilon)), \ x_2 \geq 0\}$$

we have $\dot{x}_1 > 0$ and $\dot{x}_2 \geq 0.5$. Hence, a solution $x = x(t)$ such that $x(0) \in K$ cannot satisfy the inequality $|x(t)| < 1/(2e)$ for all $t \geq 0$.