Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.243j (Fall 2003): DYNAMICS OF NONLINEAR SYSTEMS

by A. Megretski

Problem Set 2 Solutions[1]

Problem 2.1

Consider the feedback system with external input $r = r(t)$, a causal linear time invariant forward loop system G with input $u = u(t)$, output $v = v(t)$, and impulse response $g(t) = 0.1\delta(t) + (t+\bar{a})^{-1/2}e^{-t}$, where $\bar{a} \geq 0$ is a parameter, and a memoryless nonlinear feedback loop $u(t) = r(t) + \phi(v(t))$, where $\phi(y) = \sin(y)$. It is customary to require *well-posedness* of such feedback models,

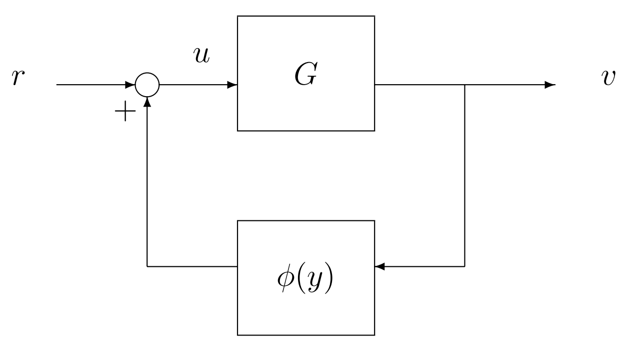

Figure 2.1: Feedback setup for Problem 2.1

which will usually mean existence and uniqueness of solutions $v = v(t)$, $u = u(t)$ of system equations

$$v(t) = 0.1u(t) + \int_0^t h(t-\tau)u(\tau)d\tau, \quad u(t) = r(t) + \phi(v(t))$$

on the time interval $t \in [0, \infty)$ for every bounded input signal $r = r(t)$.

[1] Version of October 8, 2003

(a) Show how Theorem 3.1 from the lecture notes can be used to prove well-posedness in the case when $\bar{a} > 0$.

In terms of the new signal variable

$$y(t) = v(t) - 0.1\phi(v(t)) - 0.1r(t)$$

system equations can be re-written as

$$y(t) = \int_0^t h(t-\tau)[r(\tau) + \theta(y(\tau) + 0.1r(\tau))]d\tau,$$

where

$$h(t) = \begin{cases} (t+a)^{-1/2}e^{-t}, & t \geq 0 \\ 0, & \text{otherwise,} \end{cases}$$

and $\theta : \mathbf{R} \mapsto \mathbf{R}$ is the function which maps $z \in \mathbf{R}$ into $\phi(q)$, with q being the solution of

$$q - 0.1\phi(q) = z.$$

Since ϕ is continuously differentiable, and its derivative ranges in $[-1, 1]$, θ is continuously differentiable as well, and its derivative ranges between $1/1.1$ and $1/0.9$.

For every constant $T \in [0, \infty)$, the equation for $y(t)$ with $t \geq T$ can be re-written as

$$y(t) = y(T) + \int_T^t a_T(y(\tau), \tau, t)d\tau,$$

where

$$a_T(\bar{y}, \tau, t) = h(t-\tau)[r(\tau) + \theta(y(\tau) + 0.1r(\tau))] + h_T(t),$$

$$h_T(t) = \int_0^T \dot{h}(t-\tau)[r(\tau) + \theta(y(\tau) + 0.1r(\tau))]d\tau.$$

When parameter $\bar{a}$ takes a positive value, function $a = a_T$ satisfies conditions of Theorem 3.1 with $X = \mathbf{R}^n$, $\bar{x}_0 = y(T)$, $r = 1$, and $t_0 = T$, with $K = K(\bar{a})$ being a function of $\bar{a} \neq 0$, and

$$M = M_T = M_0(a)(1 + \max_{t\in[0,T]} |y(t)|).$$

Hence a solution $y = y(\cdot)$ defined on an interval $t \in [0, T]$ can be extended in a unique way to the interval $t \in [0, T_+]$, where

$$T_+ - T = \min\{1/M_T, 1/(2K)\},$$

and

$$\max_{t\in[0,T_+]} |y(t)| \leq M_T(T_+ - T) + \max_{t\in[0,T]} |y(t)|).$$

Starting with $T = T(0) = 0$, for $k = 0, 1, 2, \dots$ define $T(k+1)$ as the T_+ calculated for $T = T(k)$. To finish the proof of well posedness, we have to show that $T(k) \to \infty$ as $k \to \infty$. Indeed, since

$$M_{T(k)}(T(k+1) - T(k)) = M_{T(k)} \min\{1/M_{T(k)}, 1/(2K)\} \leq 1,$$

$M_{T(k)}$ grows not faster than linearly with k. Hence $T(k+1) - T(k)$ decrease not faster than c/k, and therefore $T(k) \to \infty$ as $k \to \infty$.

(b) PROPOSE A GENERALIZATION OF THEOREM 3.1 WHICH CAN BE APPLIED WHEN $\bar{a} = 0$ AS WELL.

An appropriate generalization, relying on integral time-varying bounds for a and its increments, rather than their maximal values, is suggested at the end of proof of Theorem 3.1 in the lecture notes.

Problem 2.2

READ THE SECTION OF LECTURE 4 HANDOUTS ON LIMIT SETS OF TRAJECTORIES OF ODE (IT WAS *not* COVERED IN THE CLASSROOM).

(a) GIVE AN EXAMPLE OF A CONTINUOUSLY DIFFERENTIABLE FUNCTION $a:\ \mathbf{R}^2 \mapsto \mathbf{R}^2$, AND A SOLUTION OF ODE

$$\dot{x}(t) = a(x(t)), \tag{2.1}$$

FOR WHICH THE LIMIT SET CONSISTS OF A SINGLE TRAJECTORY OF A NON-PERIODIC AND NON-EQUILIBRIUM SOLUTION OF (2.1).

The limit trajectory should be that of a maximal solution $x:\ (t_1, t_2) \mapsto \mathbf{R}^2$ such that $|x(t)| \to \infty$ as $t \to t_1$ or $t \to t_2$.

To construct a system with such limit trajectory, start with a planar ODE for which every solution, except the equilibrium solution at the origin, converges to a periodic solution which trajectory is the unit circle. Considering $\mathbf{R}^2$ as the set of all complex numbers, one such ODE can be written as

$$\dot{z}(t) = (1 - |z(t)| + j)|z(t)|z(t), \text{ where } j = \sqrt{-1},$$

where every solution with $z(0) \neq 0$ converges to the trajectory of periodic solution $z_0(t) = e^{jt}$. Now apply the substitution

$$z = \frac{1}{w} + 1,$$

which moves the point $z = 1$ to $w = \infty$ (and also moves $z = \infty$ to $w = 0$). For the resulting system

$$\dot{w}(t) = -w(t)(1 + w(t))(1 + j - |(1 + w(t))/w(t)|)|(1 + w(t))/w(t)|, \tag{2.2}$$

every solution $w(\cdot)$ with $w(0) \neq 0$ will have the straight line passing through the points $w = -1/2$ and $w = 1/(j-1)$ (trajectory of the solution $w_0(t) = 1/(e^{jt} - 1)$, defined for $t \in (0, 2\pi)$), as its limit set. However, the right side of (2.2) is not a continuously differentiable function of w: there is a discontinuity at $w = 0$. To fix this problem, multiply the right side by the real number $|w(t)|^4$, which yields

$$a(w) = -w(1+w)((1+j)|w|^2 - |(1+w)w|)|(1+w)w|.$$

For the resulting system, every trajectory except the equilibrium at $w = 0$ has the same limit set as defined before.

(b) GIVE AN EXAMPLE OF A CONTINUOUSLY DIFFERENTIABLE FUNCTION $a: \mathbf{R}^n \mapsto \mathbf{R}^n$, AND A *bounded* SOLUTION OF ODE (2.1), FOR WHICH THE LIMIT SET CONTAINS NO EQUILIBRIA AND NO TRAJECTORIES OF PERIODIC SOLUTIONS.

It is possible to do this with a 4th order linear time-invariant system with purely imaginary poles:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= -x_1(t), \\ \dot{x}_3(t) &= \pi x_4(t), \\ \dot{x}_4(t) &= -\pi x_3(t). \end{aligned}$$

The solution

$$x(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \\ \sin(\pi t) \\ \cos(\pi t) \end{bmatrix}$$

of this ODE has the limit set

$$\Omega = \left\{ \begin{bmatrix} \sin(t_1) \\ \cos(t_1) \\ \sin(t_2) \\ \cos(t_2) \end{bmatrix} : \quad t_1, t_2 \in \mathbf{R} \right\}.$$

Indeed, since π is not a rational number, every real number can be approximated arbitrarily well by $2\pi k - 2q$ where k, q are arbitrarily large positive integers. Hence the difference between $t_1 + 2\pi k$ and $t_2/\pi + 2q$ can be made arbitrarily small for every given pair $t_1, t_2 \in \mathbf{R}$. For $t = t_1 + 2\pi k$ this implies that

$$\sin(t) = \sin(t_1),\ \cos(t) = \cos(t_1),\ \sin(\pi t) \approx \sin(t_2 + 2\pi q) = \sin(t_2),\ \ \cos(\pi t) \approx \cos(t_2).$$

Every solution with $x(0)$ in Ω has the form

$$x(t) = \begin{bmatrix} \sin(t+t_1) \\ \cos(t+t_1) \\ \sin(\pi t+t_2) \\ \cos(\pi t+t_2) \end{bmatrix},$$

and hence is not periodic.

An example with $n = 3$ is also possible. However, such example would require more work, since it cannot be given by a linear system.

(c) Use Theorem 4.3 from the lecture notes to derive the Poincare-Bendixon theorem: *if a set $X \subset \mathbf{R}^2$ is compact (i.e. closed and bounded), positively invariant for system (2.1) (i.e. $x(t,\bar{x}) \in X$ for all $t \geq 0$ and $\bar{x} \in X$), and contains no equilibria, then the limit set of every solution starting in X is a closed orbit (i.e. the trajectory of a periodic solution).* Assume that $a: \mathbf{R}^2 \mapsto \mathbf{R}^2$ is continuously differentiable.

Let $x_0 : (t_1, t_2) \mapsto \mathbf{R}^2$ be a maximal solution of (2.1) such that $t_1 < 0 < t_2$ and $x(0) \in X$. Then, by the invariance of X, $x(t) \in X$ for all $t \geq 0$. Hence $x(t)$ is bounded for $t \geq 0$, and hence $t_2 = \infty$. Appllying Theorem 4.3 to x_0, note first that scenario (a) cannot take place (since $x(t)$ is bounded for $t \geq 0$). On the other hand, scenario (c) also cannot take place. Indeed, otherwise let $x_1 : (t_1^1, t_2^1) \mapsto \mathbf{R}^2$ be a maximal solution of (2.1) such that $x_1(t)$ is a limit point of $x_0(\cdot)$ for all $t \in (t_1^1, t_2^1)$. Since X is closed and $x_0(t) \in X$ for $t \geq 0$, all limit points of x_0 lie in X. Hence $x_1(t)$ is in X, and $t_2^1 = \infty$. According to scenario (c), the limit

$$\bar{x} = \lim_{t\to\infty} x_1(t)$$

exists, which implies $a(\bar{x}) = 0$, contradicting the assumptions. Hence only scenario (b) takes place, which is what we had to prove.

Problem 2.3

Use the index theory to prove the following statements.

(a) If $n > 1$ is even and $F : S^n \mapsto S^n$ is continuous then there exists $x \in S^n$ such that $x = F(x)$ or $x = -F(x)$.

Assume, to the contrary, that $x \neq F(x)$ and $-x \neq F(x)$ for all $x \in S^n$. Then

$$H(x,t) = \frac{(2t-1)x + t(1-t)F(x)}{|(2t-1)x + t(1-t)F(x)|}$$

is a continuous homotopy between $H(x,0) = -x$ and $H(x,1) = x$. Since index of the map $x \mapsto -x$ equals $(-1)^{n+1}$, and index of the map $x \mapsto x$ equals 1, a contradiction results.

(b) The equations for the harmonically forced nonlinear oscillator

$$\ddot{y}(t) + \dot{y}(t) + (1 + y(t)^2)y(t) = 100\cos(t)$$

have at least one 2π-periodic solution. **Hint:** Show first that, for

$$V(t) = \dot{y}(t)^2 + y(t)^2 + y(t)\dot{y}(t) + 0.5y(t)^4,$$

the inequality

$$\dot{V}(t) \leq -c_1 V(t) + c_2,$$

where c_1, c_2 are some positive constants, holds for all t.

Differentiating $V(t)$ along a system solution $y = y(t)$ yields, for $w(t) = 100\cos(t)$,

$$\begin{aligned}
\dot{V} &= -y^2 - y\dot{y} - \dot{y}^2 - y^4 + 2(\dot{y} + y/2)w \\
&= -0.5V - 0.5(\dot{y} + y/2)^2 + 2(\dot{y} + y/2)w - 3/8y^2 \\
&= -0.5V + 2w^2 - 0.5(\dot{y} + y/2 + 2w)^2 - 3/8y^2 \\
&\leq -0.5V + 20000.
\end{aligned}$$

Hence the derivative of

$$r(t) = e^{0.5t}(V(t) - 40000)$$

is non-positive at all times, i.e. $r = r(t)$ is monotonically non-increasing.

Consider the function $G_0 : \mathbf{R}^2 \mapsto \mathbf{R}^2$ which maps the vector of initial conditions $x(0) = [y(0); \dot{y}(0)]$ to the vector $x(T) = [y(T); \dot{y}(T)]$, where $T = 2\pi k$ and $k > 0$ is an integer parameter to be chosen later. By continuity of dependence of solutions of ODE on parameters, G_0 is continuous. Also, since

$$V(t) \leq 3|x(t)|^2 + 0.5|x(t)|^4 \leq |x(t)|^4 + 5,$$

it follows that

$$e^{0.5T}(V(T) - 40000) \leq V(0) - 40000 \leq |x(0)|^4,$$

which implies

$$V(T) \leq 40000 + e^{-\pi k}|x(0)|^4.$$

Since $V(t) \geq 0.5|x(t)|^2$, it follows that

$$|x(T)| \leq 80000 + 2e^{-\pi k}(|x(0)|^4 - 39995).$$

Hence, if $|x(0)| \leq 300$ and

$$k \geq \frac{\log(2) + 4\log(30)}{\pi} \approx 4.55$$

then $|x(T)| \leq 300$.

Now consider the function $G: \ B^2 \mapsto B^2$, where B^2 is the unit ball in $\mathbf{R}^2$, defined by

$$G(\bar{x}) = G_0(300\bar{x})/300.$$

The function satisfies the conditions of the Brower's fixed point theorem, and hence there exists $\bar{x} \in B^2$ such that $G(\bar{x}) = \bar{x}$. By the definition of G, the solution of the nonlinear oscillator equations with

$$\begin{bmatrix} y(0) \\ \dot{y}(0) \end{bmatrix} = 300\bar{x}$$

will be periodic with period $T = 10\pi$.