Massachusetts Institute of Technology

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Problem Set 1 Solutions[1]

Problem 1.1

BEHAVIOR SET $\mathcal{B}$ OF AN AUTONOMOUS SYSTEM WITH A SCALAR BINARY DT OUTPUT CONSISTS OF ALL DT SIGNALS $w = w(t) \in \{0, 1\}$ WHICH CHANGE VALUE AT MOST ONCE FOR $0 \le t < \infty$.

(a) GIVE AN EXAMPLE OF TWO SIGNALS $w_1, w_2 \in \mathcal{B}$ WHICH COMMUTE AT $t = 3$, BUT DO NOT DEFINE SAME STATE OF $\mathcal{B}$ AT $t = 3$.

To answer this and the following questions, let us begin with formulating necessary and sufficient conditions for two signals $z_1, z_2 \in \mathcal{B}$ to commute and to define same state of $\mathcal{B}$ at a given time t.

For $w \in \mathcal{B}$, $t \in [0, \infty)$ let

$$w[t] = \begin{cases} \lim_{\tau\to t, \tau<t} w(\tau), & \text{if } t > 0, \\ w(0), & \text{if } t = 0 \end{cases}$$

be the left side limit value of w at t. Let

$$N_+(w, t) = \begin{cases} 0, & \text{if } w(t) = \lim_{\tau\to\infty} w(\tau), \\ 1, & \text{otherwise} \end{cases}$$

be the number of discontinuities of $w(\tau)$ between $\tau = t$ and $\tau = \infty$. Similarly, let

$$N_-(w, t) = \begin{cases} 0, & \text{if } w(0) = w[t], \\ 1, & \text{otherwise} \end{cases}$$

be the number of discontinuities of $w(\tau)$ between $\tau = 0$ and $\tau = t - 0$.

[1] Version of October 2, 2003

Lemma 1.1 *Signals* $z_1, z_2 \in \mathcal{B}$ *commute at time* $t \in [0, \infty)$ *if and only if* $z_1(t) = z_2(t)$ *and*

$$N_-(z_1, t) + N_+(z_2, t) + |z_2(t) - z_1[t]| \leq 1 \tag{1.1}$$

and

$$N_-(z_2, t) + N_+(z_1, t) + |z_1(t) - z_2[t]| \leq 1. \tag{1.2}$$

Proof First note that the "hybrid" signal z_{12}, obtained by "gluing" the past of z_1 (before time t) to the future of z_2 (from t to ∞), is a discrete time signal if and only if $z_1(t) = z_2(t)$. Moreover, since he discontinuities of z_{12} result from three causes: discontinuities of $z_1(\tau)$ before $\tau = t$, discontinuities of z_2 between $\tau = t$ and $\tau = \infty$, and the inequality between $z_1[t]$ and $z_1(t)$, condition (1.1) is necessary and sufficient for $z_{12} \in \mathcal{B}$ (subject to $z_1(t) = z_2(t)$). Similarly, considering the discontinuities of the other "hybrid" obtained by "gluing" the past of z_2 to the future of z_2 yields (1.2). ■

It follows immediately from Lemma 1.1 that signals $z_1, z_2 \in \mathcal{B}$ define same state of $\mathcal{B}$ at time $t \in [0, \infty)$ if and only if

$$N_-(z_1, t) = N_-(z_2, t),\ D(z_1, t) = D(z_2, t),\ N_+(z_1, t) = N_+(z_2, t),\ z_1(t) = z_2(t), \tag{1.3}$$

where for $w \in \mathcal{B}$

$$D(w, t) = |w(t) - w[t]|$$

is the indicator of a discontinuity at t.

For $k \in \mathbf{Z}_+$ let $u_k \in \mathcal{B}$ be defined by

$$u_k(t) = \begin{cases} 0, & t < k, \\ 1, & t \geq k. \end{cases}$$

Then u_1 and u_0 commute but do not define same state of $\mathcal{B}$ at time $t = 3$.

(b) GIVE AN EXAMPLE OF TWO *different* SIGNALS $w_1, w_2 \in \mathcal{B}$ WHICH DEFINE SAME STATE OF $\mathcal{B}$ AT $t = 4$.

u_1 and u_2.

(c) FIND A TIME-INVARIANT DISCRETE-TIME FINITE STATE-SPACE "DIFFERENCE INCLUSION" MODEL FOR $\mathcal{B}$, I.E. FIND A *finite* SET X AND FUNCTIONS $g: X \mapsto \{0, 1\}$, $f: X \mapsto S(X)$, WHERE $S(X)$ DENOTES THE SET OF ALL NON-EMPTY SUBSETS OF X, SUCH THAT A SEQUENCE $w(0), w(1), w(2), \ldots$ CAN BE OBTAINED BY SAMPLING A SIGNAL $w \in \mathcal{B}$ IF AND ONLY IF THERE EXISTS A SEQUENCE $x(0), x(1), x(2), \ldots$ OF ELEMENTS FROM X SUCH THAT

$$x(t+1) \in f(x(t)) \quad \text{and} \quad w(t) = g(x(t)) \quad \text{for } t = 0, 1, 2, \ldots.$$

(Figuring out which pairs of signals define same state of $\mathcal{B}$ at a given time is one possible way to arrive at a solution.)

Condition (1.3) naturally calls for X to be the set of all possible combinations

$$x(t) = [N_-(w,t); N_+(w,t); D(w,t); w(t)].$$

Note that not more than one of the first three components can be non-zero at a given time instance, and hence the total number of possible values of $x(t)$ is eight, which further reduces to four at $t = 0$, since

$$N_-(w,0) = D(w,0) = 0 \;\; \forall\, w \in \mathcal{B}.$$

The dynamics of $x(t)$ is given by

$$f([0;0;0;0]) = \{[0;0;0;0]\},$$

$$\begin{aligned}
f([0;0;0;1]) &= \{[0;0;0;1]\},\\
f([1;0;0;0]) &= \{[1;0;0;0]\},\\
f([1;0;0;1]) &= \{[1;0;0;1]\},\\
f([0;1;0;0]) &= \{[1;0;0;0]\},\\
f([0;1;0;1]) &= \{[1;0;0;1]\},\\
f([0;0;1;0]) &= \{[0;0;1;0],[0;1;0;1]\},\\
f([0;0;1;1]) &= \{[0;0;1;1],[0;1;0;0]\},
\end{aligned}$$

while $g(x(t))$ is simply the last bit of $x(t)$.

This model is not the *minimal* state space model of $\mathcal{B}$. Note that last two bits of $x(t+1)$, as well as $w(t)$, depend only on the last two bits of $x(t)$. Hence a model of $\mathcal{B}$ with a two-bit state space $X_* = \{0,1\} \times \{0,1\}$ can be given by

$$f_*([0;0]) = \{[0;0]\},\; f_*([0;1]) = \{[0;1]\},\; f_*([1;0]) = \{[0;1],[1;0]\},\; f_*([1;1]) = \{[0;0],[1;1]\},$$

and

$$g_*([x_1;x_2]) = x_2.$$

Problem 1.2

Consider differential equation

$$\ddot{y}(t) + \text{sgn}(\dot{y}(t) + y(t)) = 0.$$

(a) Write down an equivalent ODE $\dot{x}(t) = a(x(t))$ for the state vector $x(t) = [y(t); \dot{y}(t)]$.

$$a\left(\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}\right) = \begin{bmatrix} \bar{x}_2 \\ -\text{sgn}(\bar{x}_1 + \bar{x}_2) \end{bmatrix}.$$

(b) Find all vectors $\bar{x}_0 \in \mathbf{R}^2$ for which the ODE from (a) does not have a solution $x : [t_0, t_1] \mapsto \mathbf{R}^2$ (with $t_1 > t_0$) satisfying initial condition $x(t_0) = x_0$.

Solutions (forward in time) do not exist for

$$\bar{x}_0 \in X_0 = \left\{ \begin{bmatrix} \bar{x}_{01} \\ \bar{x}_{02} \end{bmatrix} \in \mathbf{R}^2 : \ \bar{x}_{01} + \bar{x}_{02} = 0, \ \bar{x}_{01} \in [-1, 1], \ \bar{x}_{01} \neq 0 \right\}.$$

To show this, note first that, for

$$\bar{x}_{01} + \bar{x}_{02} \geq 0, \ \bar{x}_{02} > 1,$$

a solution is given by

$$x(t) = \begin{bmatrix} \bar{x}_{01} + t\bar{x}_{02} - t^2/2 \\ \bar{x}_{02} - t \end{bmatrix}, \quad t \in [0, 2(\bar{x}_{02} - 1)].$$

Similarly, for

$$\bar{x}_{01} + \bar{x}_{02} \leq 0, \ \bar{x}_{02} < -1,$$

a solution is given by

$$x(t) = \begin{bmatrix} \bar{x}_{01} + t\bar{x}_{02} + t^2/2 \\ \bar{x}_{02} + t \end{bmatrix}, \quad t \in [0, 2(-\bar{x}_{02} - 1)].$$

Finally, for $\bar{x}_0 = 0$ there is the equilibrium solution $x(t) \equiv 0$.

Now it is left to prove that no solutions with $x(0) \in X_0$ exist. Assume that, to the contrary, $x : [0, \epsilon] \mapsto \mathbf{R}$ is a solution with $\epsilon > 0$ and $x(0) = [-t, t]$ for some $t \in [-1, 1]$, $t \neq 0$. Without loss of generality, assume that $0 < t \leq 1$.

Since x is continuous, there exist $\delta \in (0, \epsilon)$ such that $x_2(t) > 0$ for all $t \in [0, \delta]$. Let t_0 be the argument of minimum of $x_1(t) + x_2(t)$ for $t \in [0, \delta]$. If $x_1(t_0) + x_2(t_0) < 0$ then $x_2(t) - \text{sgn}(x_1(t) + x_2(t)) \geq 1$ for t in a neigborhood of t_0, which contradicts the assumption that t_0 is an argument of a minimum. Hence $x_1(t)+x_2(t) \geq 0$ for all $t \in [0,\delta]$. Moreover, since x_1 is an integral of $x_2 > 0$, $x_1(t)$ is strictly monotonically non-increasing on $[0,\delta]$, and hence $x_1(t) > -1$ for all $t \in (0,\delta]$.

Let t_0 be the argument of maximum of $x_1(t)+x_2(t)$ on $[0,\delta]$. If $x_1(t_0)+x_2(t_0) > 0$ then $x_1(t)+x_2(t) > 0$ in a neigborhood of t_0. Combined with $x_1(t) > -1$, this yields

$$d(t) = x_2(t) - \text{sgn}(x_1(t)+x_2(t)) < -x_1(t) - \text{sgn}(x_1(t)+x_2(t)) < 1-1 = 0.$$

Since $x_1 + x_2$ is an integral of d, this contradicts the assumption that t_0 is an argument of a maximum. Hence $x_1(t)+x_2(t) = 0$ for $t \in [0,\delta]$, which implies that $x_2(t)$ is a constant. Hence $x_1(t)$ is a constant as well, which contradicts the strict monotonicity of $x_1(t)$.

(c) Define a semicontinuous convex set-valued function $\eta : \mathbf{R}^2 \mapsto 2^{\mathbf{R}^2}$ such that $a(\bar{x}) \in \eta(\bar{x})$ for all x. Make sure the sets $\eta(\bar{x})$ are the smallest possible subject to these constraints.

First note that $a([\bar{x}_1,\bar{x}_2])$ converges to $[\bar{x}_2^0; 1]$ as $\bar{x}_2 \to \bar{x}_2^0$ within the open half plane $\bar{x}_1+\bar{x}_2 < 0$. Similarly, $a([\bar{x}_1,\bar{x}_2])$ converges to $[\bar{x}_2^0; -1]$ as $\bar{x}_2 \to \bar{x}_2^0$ subject to $\bar{x}_1+\bar{x}_2 > 0$. Hence one must have $\eta(\bar{x}) \supset \eta_0(\bar{x})$, where

$$\eta_0\left(\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}\right) = \left\{\begin{bmatrix} \bar{x}_2 \\ -t \end{bmatrix} : \ t \in \nu(\bar{x}_1+\bar{x}_2)\right\},$$

$$\nu(y) = \begin{cases} \{1\}, & y > 0, \\ \{-1\}, & y < 0, \\ \text{[-1,1]}, & y = 0. \end{cases}.$$

On the other hand, it is easy to check that the compact convex set-valued function η_0 is semicontinuous. Hence $\eta = \eta_0$.

(d) Find explicitly all solutions of the differential inclusion $\dot{x}(t) \in \eta(x(t))$ satisfying initial conditions $x(0) = x_0$, where x_0 are the vectors found in (b). Such solutions are calles *sliding modes*.

The proof in (b) can be repeated to show that all such solutions will stay on the hyperplane $x_1(t)+x_2(t) = 0$. Hence

$$x_1(t) = x_1(0)e^{-t}, \quad x_2(t) = x_2(0)e^{-t}.$$

(e) Repeat (c) for $a : \mathbf{R}^2 \mapsto \mathbf{R}^2$ defined by

$$a([x_1; x_2]) = [\text{sgn}(x_1); \text{sgn}(x_2)].$$

$$\eta\left(\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}\right) = \left\{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} : \ c_1 \in \nu(\bar{x}_1), \ c_2 \in \nu(\bar{x}_2)\right\}.$$

Problem 1.3

For the statements below, state whether they are true or false. For true statements, give a *brief* proof (can refer to lecture notes or books). For false statements, give a counterexample.

(a) All maximal solutions of ODE $\dot{x}(t) = \exp(-x(t)^2)$ are defined on the whole time axis $\{t\} = \mathbf{R}$.

This statement is **true**. Indeed, a maximal solution $x = x(t)$ is defined on an interval with a finite bound t_* only when $|x(t)| \to \infty$ as $t \to t_*$. However, $x(t)$ is an integral of a function not exceeding 1 by absolute value. Hence $|x(t) - x(t_0)| \leq |t - t_0|$ for all t, and therefore $|x(t)|$ cannot approach infinity on a finite time interval.

(b) All solutions $x: \mathbf{R} \mapsto \mathbf{R}$ of the ODE

$$\dot{x}(t) = \begin{cases} x(t)/t, & t \neq 0, \\ 0, & t = 0 \end{cases}$$

are such that $x(t) = -x(-t)$ for all $t \in \mathbf{R}$.

This statement is **false**. Indeed, for every pair $c_1, c_2 \in \mathbf{R}$ the function

$$x(t) = \begin{cases} c_1 t, & t \leq 0, \\ c_2 t, & t > 0 \end{cases}$$

is a solution of the ODE, which can be verified by checking that

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} \frac{x(t)}{t} dt \quad \forall\, t_1, t_2.$$

(c) If constant signal $w(t) \equiv 1$ belongs to a system behavior set $\mathcal{B}$, but constant signal $w(t) \equiv -1$ does not then the system is not linear.

This statement is **true**. Indeed, if $\mathcal{B}$ is linear then $cw \in \mathcal{B}$ for all $c \in \mathbf{R}$, $w \in \mathcal{B}$. With $c = -1$ this means that, for a linear system, $w \in \mathcal{B}$ if and only if $-w \in \mathcal{B}$.