Massachusetts Institute of Technology

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6.243j (Fall 2003): DYNAMICS OF NONLINEAR SYSTEMS

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Problem Set 8[1]

Problem 8.1

Autonomous system equations have the form

$$\ddot{y}(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}' Q \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}, \tag{8.1}$$

where y is the scalar output, and $Q = Q'$ is a given symmetric 2-by-2 matrix with real coefficients.

(a) Find all Q for which there exists a C^∞ bijection $\psi : \mathbf{R}^2 \mapsto \mathbf{R}^2$, matrices A, C, and a C^∞ function $\phi : \mathbf{R} \mapsto \mathbf{R}^2$ such that $z = \psi(y, \dot{y})$ satisfies the ODE

$$\dot{z}(t) = Az(t) + \phi(y(t)), \quad y(t) = Cz(t)$$

whenever $y(\cdot)$ satisfies (8.1).

(b) For those Q found in (a), construct C^∞ functions $F = F_Q : \mathbf{R}^2 \times \mathbf{R} \mapsto \mathbf{R}^2$ and $H = H_Q : \mathbf{R}^2 \mapsto \mathbf{R}$ such that $H_Q(\eta(t)) - \dot{y}(t) \to 0$ as $t \to \infty$ whenever $y : [0, \infty) \mapsto \mathbf{R}$ is a solution of (8.1), and

$$\dot{\eta}(t) = F_Q(\eta(t), y(t)).$$

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Problem 8.2

A linear constrol system

$$\begin{cases} \dot{x}_1(t) &= x_2(t) + w_1(t), \\ \dot{x}_2(t) &= -x_1(t) - x_2(t) + u + w_2(t) \end{cases}$$

is equipped with the nonlinear sensor

$$y(t) = x_1(t) + \sin(x_2(t)) + w_3(t),$$

where $w_i(\cdot)$ represent plant disturbances and sensor noise satisfying a uniform bound $|w_i(t)| \leq d$. Design an observer of the form

$$\dot{\eta}(t) = F(\eta(t), y(t), u(t))$$

and constants $d_0 > 0$ and $C > 0$ such that

$$|\eta(t) - x(t)| \leq Cd \quad \forall\, t \geq 0$$

whenever $\eta(0) = x(0)$ and $d < d_0$. (Try to make d_0 as large as possible, and C as small as possible.)

Problem 8.3

Is it true or false that the set $\Omega = \Omega_F = \{P\}$ of positive definite quadratic forms $V_P(\bar{x}) = \bar{x}'P\bar{x}$, where $P = P' > 0$, which are valid control Lyapunov function for a given ODE model

$$\dot{x}(t) = F(x(t), u(t)),$$

in the sense that

$$\inf_{\bar{u} \in \mathbf{R}} x'PF(\bar{x}, \bar{u}) \leq -|\bar{x}|^2 \quad \forall\, \bar{x} \in \mathbf{R}^n,$$

is linearly connected for all continuously differentiable functions $F: \mathbf{R}^n \times \mathbf{R} \mapsto \mathbf{R}^n$? (Remember that a set Ω of matrices is called linearly connected if for every two matrices $P_0, P_1 \in \Omega$ there exists a continuous function $p: [0,1] \mapsto \Omega$ such that $p(0) = P_0$ and $p(1) = P_1$. In particular, the empty set is linearly connected.)