Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.243j (Fall 2003): DYNAMICS OF NONLINEAR SYSTEMS

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Problem Set 2[1]

Problem 2.1

Consider the feedback system with external input $r = r(t)$, a causal linear time invariant forward loop system G with input $u = u(t)$, output $v = v(t)$, and impulse response $g(t) = 0.1\delta(t) + (t + a)^{-1/2}e^{-t}$, where $a \geq 0$ is a parameter, and a memoryless nonlinear feedback loop $u(t) = r(t) + \phi(v(t))$, where $\phi(y) = \sin(y)$. It is customary to require *well-posedness* of such feedback models, which will usually mean existence and uniqueness of solutions $v = v(t)$, $u = u(t)$ of system equations

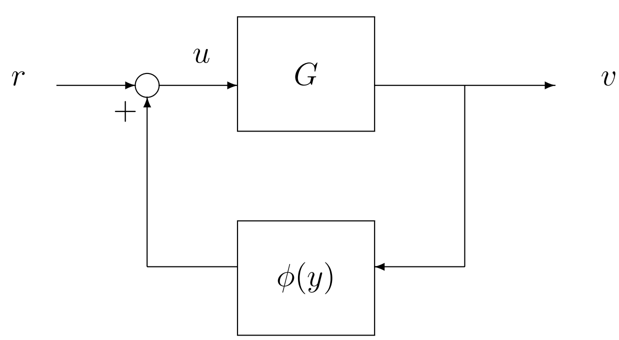

Figure 2.1: Feedback setup for Problem 2.1

$$v(t) = 0.1u(t) + \int_0^t h(t-\tau)u(\tau)d\tau, \quad u(t) = r(t) + \phi(v(t))$$

on the time interval $t \in [0, \infty)$ for every bounded input signal $r = r(t)$.

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(a) Show how Theorem 3.1 from the lecture notes can be used to prove well-posedness in the case when $a > 0$. **Hint:** it may be a good idea to begin with getting rid of the algebraic part of the system equations by introducing a new signal $e(t) = v(t) - 0.1\phi(v(t)) - 0.1r(t)$.

(b) Propose a generalization of Theorem 3.1 which can be applied when $a = 0$ as well. (You are not required to write down the proof of your generalization, but make every effort to ensure the statement is correct.)

Problem 2.2

Read the section of Lecture 4 handouts on limit sets of trajectories of ODE (it was *not* covered in the classroom).

(a) Give an example of a continuously differentiable function $a : \mathbf{R}^2 \mapsto \mathbf{R}^2$, and a solution of ODE

$$\dot{x}(t) = a(x(t)), \tag{2.1}$$

for which the limit set consists of a single trajectory of a non-periodic and non-equilibrium solution of (2.1).

(b) Give an example of a continuously differentiable function $a : \mathbf{R}^n \mapsto \mathbf{R}^n$, and a *bounded* solution of ODE (2.1), for which the limit set contains no equilibria and no trajectories of periodic solutions. **Hint:** it is possible to do this with a 4th order linear time-invariant system with purely imaginary poles.

(c) Use Theorem 4.3 from the lecture notes to derive the Poincare-Bendixon theorem: *if a set $X \subset \mathbf{R}^2$ is compact (i.e. closed and bounded), positively invariant for system (2.1) (i.e. $x(t, \bar{x}) \in X$ for all $t \geq 0$ and $\bar{x} \in X$), and contains no equilibria, then the limit set of every solution starting in X is a closed orbit (i.e. the trajectory of a periodic solution).* Assume that $a : \mathbf{R}^2 \mapsto \mathbf{R}^2$ is continuously differentiable.

Problem 2.3

Use the index theory to prove the following statements.

(a) If $n > 1$ is even and $F : S^n \mapsto S^n$ is continuous then there exists $x \in S^n$ such that $x = F(x)$ or $x = -F(x)$.

(b) The equations for the harmonically forced nonlinear oscillator

$$\ddot{y}(t) + \dot{y}(t) + (1 + y(t)^2)y(t) = 100\cos(t)$$

have at least one 2π-periodic solution. **Hint:** Show first that, for

$$V(t) = \dot{y}(t)^2 + y(t)^2 + y(t)\dot{y}(t) + 0.5y(t)^2,$$

the inequality

$$\dot{V}(t) \leq -c_1 V(t) + c_2,$$

where c_1, c_2 are some positive constants, holds for all t.