

# Lectures on Dynamic Systems and Control

Mohammed Dahleh    Munther A. Dahleh    George Verghese  
Department of Electrical Engineering and Computer Science  
Massachusetts Institute of Technology<sup>1</sup>

## Chapter 21

# Robust Performance and Introduction to the Structured Singular Value Function

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### 21.1 Introduction

As discussed in Lecture 20, a process is better described in terms of a set of plants centered around a nominal model. The robust stabilization problem is concerned with finding non conservative conditions on the stable nominal closed loop system that guarantee the stability of all possible closed loop systems. An equally important problem is the robust performance problem which is concerned with finding non conservative conditions on the nominal closed loop system that guarantee that the performance is met for all possible closed loop systems.

### 21.2 Robust Disturbance Rejection

We will focus our discussion on one prototype problem, namely, the robust disturbance rejection problem shown in Figure 21.1. This motivates the following problem:

#### Robust Disturbance Rejection Problem (RP)

Find conditions on the nominal closed-loop system  $(P_o, K)$  such that

1.  $K$  robustly stabilizes all  $P \in \Omega$ , where  $\Omega = \{P \mid P = (I + \Delta_1 W_1)P_o, \quad \|\Delta\|_\infty < 1\}$ .
2.  $\|(I + PK)^{-1}W_2\|_\infty \leq 1$  for all  $P \in \Omega$ .

From Lecture 20, a performance objective in terms of the  $\mathcal{H}_\infty$ -norm of some closed loop map between some exogenous input  $w$ , to a regulated variable  $z$ , is mathematically equivalent to a robust stabilization problem with a perturbation block mapping the regulated output  $z$  to the exogenous input  $w$ . Obviously, the new perturbed system is stable if and only if  $\|T_{zw}\|_\infty \leq 1$ , which is the performance

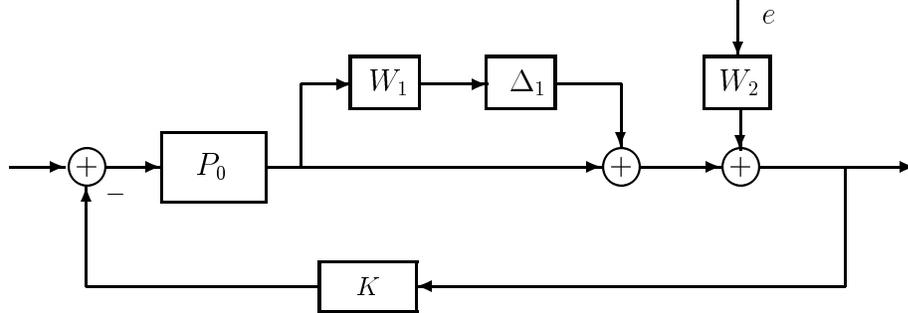


Figure 21.1: Uncertain Plant with Disturbance

objective. Notice that if the performance objective consists of several closed loop maps, then several perturbation blocks can be introduced in exactly the same fashion.

Proceeding for **RP**, we can “wrap” a frequency-weighted perturbation from the output to the input of interest, which results in the model of Figure 21.2. Next, we can re-arrange the system into the

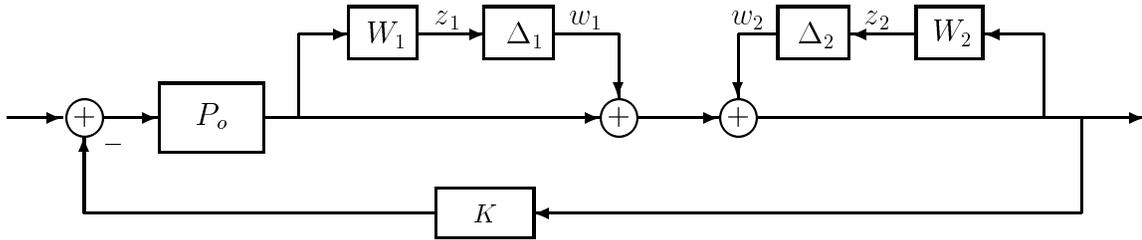


Figure 21.2: Robust Performance Model

$M$ - $\Delta$  feedback form (a nominal stable  $M$  in feedback with the perturbation  $\Delta$ ) as in Figure 21.3. In this case, however, there are multiple inputs and outputs to consider. We use the following procedure to generate  $M$  and  $\Delta$ :

1. Define  $w_i, z_i$  to be the output and input, respectively, of the perturbation  $\Delta_i$ .
2. For a total of  $m$  perturbations, compute the matrix transfer function  $M$  as the map from

$$w = \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix} \quad \text{to} \quad z = \begin{bmatrix} z_1 \\ \vdots \\ z_m \end{bmatrix}. \quad (21.1)$$

In other words, all the  $\Delta$  blocks are removed, and the transfer functions “seen” by the blocks from each input  $w_j$  to each output  $z_i$  are calculated and used as the  $(i, j)^{\text{th}}$  element of  $M$ .

3. The perturbation matrix  $\Delta$  will have the structure

$$\Delta = \begin{bmatrix} \Delta_1 & & \\ & \ddots & \\ & & \Delta_m \end{bmatrix}, \quad \|\Delta_i\|_\infty < 1. \quad (21.2)$$

For a SISO system, each  $\Delta_i(j\omega)$  is a scalar, so that  $\Delta$  becomes a diagonal matrix with complex entries. In the MIMO case,  $\Delta$  is block-diagonal.

**Example 21.1 (Robust Disturbance Rejection)**

Applying the robust performance procedure to Figure 21.2 yields:

$$M = \begin{bmatrix} -W_1(I + P_0K)^{-1}P_0K & -W_1(I + P_0K)^{-1}P_0K \\ W_2(I + P_0K)^{-1} & W_2(I + P_0K)^{-1} \end{bmatrix}. \quad (21.3)$$

The transfer functions on the diagonal are identical to those in the single-block robust stability and disturbance-rejection problems, respectively, while the off-diagonal terms account for the interaction between the two constraints. Having found the appropriate  $M$  and  $\Delta$ , we have thereby reduced the robust performance problem to a stability problem for the system of Figure 21.3.

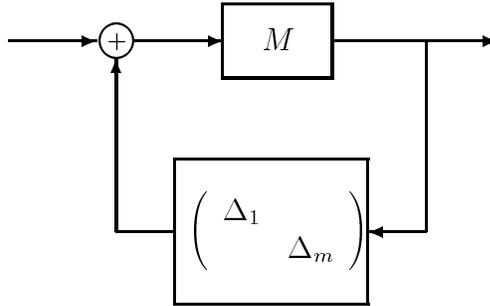


Figure 21.3:  $M$ - $\Delta$  Feedback Form

A sufficient condition for robust stability is given by the small gain theorem, namely,

$$\sigma_{\max}[M(jw)]\sigma_{\max}[\Delta(jw)] \leq \gamma < 1, \quad \text{for all } w.$$

Since  $\Delta$  is norm bounded by one, this condition translates to  $\|M\|_{\infty} \leq \gamma$ . This condition, however, is far from necessary since  $\Delta$  has a block diagonal structure.

### 21.3 The Structured Singular Value

For an unstructured perturbation, the supremum of the maximum singular value of  $M$  (*i.e.*  $\|M\|_{\infty}$ ) provides a clean and numerically tractable method for evaluating robust stability. Recall that, for the standard  $M$ - $\Delta$  loop, the system fails to be robustly stable if there exists an admissible  $\Delta$  such that  $(I - M\Delta)$  is singular. What distinguishes the current situation from the unstructured case is that we have placed constraints on the set  $\Delta$ . Given this more limited set of admissible perturbations, we desire a measure of robust stability similar to  $\|M\|_{\infty}$ . This can be derived from the *structured singular value*  $\mu(M)$ .

**Definition 21.1** The structured singular value of a complex matrix  $M$  with respect to a class of perturbations  $\Delta$  is given by

$$\mu(M) \triangleq \frac{1}{\inf\{\sigma_{\max}(\Delta) \mid \det(I - M\Delta) = 0\}}, \quad \Delta \in \Delta. \quad (21.4)$$

If  $\det(I - M\Delta) \neq 0$  for all  $\Delta \in \Delta$ , then  $\mu(M) = 0$ .

**Theorem 21.1** The  $M$ - $\Delta$  System is stable for all  $\Delta \in \Delta$  with  $\|\Delta\|_\infty < 1$  if and only if

$$\sup_{\omega} \mu(M(j\omega)) \leq 1.$$

**Proof:** Immediate, from the definition. Clearly, if  $\mu \leq 1$ , then the norm of the smallest allowable destabilizing perturbation  $\Delta$  must by definition be greater than 1.

## 21.4 Properties of the Structured Singular Value

It is important to note that  $\mu$  is a function that depends on the perturbation class  $\Delta$  (sometimes, this function is denoted by  $\mu_\Delta$  to indicate this dependence). The following are useful properties of such a function.

1.  $\mu(M) \geq 0$ .
2. If  $\Delta = \{\lambda I \mid \lambda \in \mathbb{C}\}$ , then  $\mu(M) = \rho(M)$ , the spectral radius of  $M$  (which is equal to the magnitude of the eigenvalue of  $M$  with maximum magnitude).
3. If  $\Delta = \{\Delta \mid \Delta \text{ is an arbitrary complex matrix}\}$  then  $\mu = \sigma_{\max}(M)$ , from which  $\sup_{\omega} \mu = \|M\|_\infty$ .

Property 2 shows that the spectral radius function is a particular  $\mu$  function with respect to a perturbation class consisting of matrices of the form of scaled identity. Property 3 shows that the maximum singular value function is a particular  $\mu$  function with respect to a perturbation class consisting of arbitrary norm bounded perturbations (no structural constraints).

4. If  $\Delta = \{\text{diag}(\Delta_1, \dots, \Delta_n) \mid \Delta_i \text{ complex}\}$ , then  $\mu(M) = \mu(D^{-1}MD)$  for any  $D = \text{diag}(d_1, \dots, d_n)$ ,  $|d_i| > 0$ . The set of such scales is denoted  $\mathcal{D}$ .

This can be seen by noting that  $\det(I - AB) = \det(I - BA)$ , so that  $\det(I - D^{-1}MD\Delta) = \det(I - MD\Delta D^{-1}) = \det(I - M\Delta)$ . The last equality arises since the diagonal matrices  $\Delta$  and  $D$  commute.

5. If  $\Delta = \text{diag}(\Delta_1, \dots, \Delta_n)$ ,  $\Delta_i$  complex, then  $\rho(M) \leq \mu(M) \leq \sigma_{\max}(M)$ .

This property follows from the following observation: If  $\Delta_1 \subset \Delta_2$ , then  $\mu_1 \leq \mu_2$ . It is clear that the class of perturbations consisting of scaled identity matrices is a subset of  $\Delta$  which is a subset of the class of all unstructured perturbations.

6. From 4 and 5 we have that  $\mu(M) = \mu(D^{-1}MD) \leq \inf_{D \in \mathcal{D}} \sigma_{\max}(D^{-1}MD)$ .

## 21.5 Computation of $\mu$

In general, there is no closed-form method for computing  $\mu$ . Upper and lower bounds may be computed and refined, however. In these notes we will only be concerned with computing the upper bound. If  $\Delta = \text{diag}(\Delta_1, \dots, \Delta_n)$ , then the upper bound on  $\mu$  is something that is easy to calculate. Furthermore, property 6 above suggests that by infimizing  $\sigma_{\max}(D^{-1}MD)$  over all possible diagonal scaling matrices, we obtain a better approximation of  $\mu$ . This turns out to be a convex optimization problem at each

frequency, so that by infimizing over  $\mathcal{D}$  at each frequency, the tightest upper bound over the set of  $\mathcal{D}$  may be found for  $\mu$ .

We may then ask when (if ever) this bound is tight. In other words, when is it truly a least upper bound. The answer is that for three or fewer  $\Delta$ 's, the bound is tight. The proof of this is involved, and is beyond the scope of this class. Unfortunately, for four or more perturbations, the bound is not tight, and there is no known method for computing  $\mu$  exactly for more than three perturbations.

## 21.6 Robust Disturbance Rejection (SISO)

As shown earlier, the disturbance rejection requirement could be converted to a robust stability problem with two blocks of uncertainty, as in Figure 21.2, where  $\Delta_1$  and  $\Delta_2$  are SISO stable systems. Hence  $\Delta$  is the set of  $2 \times 2$  diagonal complex matrices (which result from evaluating  $\Delta$  at each frequency).

Now, since this is a two-block problem, it should be possible to find  $\mu$  by infimizing  $\sigma_{\max}(D^{-1}MD)$ . We have  $D = \text{diag}(d_1, d_2)$ , so that

$$\mu(M(j\omega)) = \inf_{d_1, d_2 > 0} \left\{ \sigma_{\max} \left[ \underbrace{\begin{bmatrix} -\frac{W_1 P_0 K}{1 + P_0 K}(j\omega) & -\frac{d_2}{d_1} \frac{W_1 K}{1 + P_0 K}(j\omega) \\ \frac{d_1}{d_2} \frac{W_2 P_0}{1 + P_0 K}(j\omega) & \frac{W_2}{1 + P_0 K}(j\omega) \end{bmatrix}}_{A(\alpha)} \right] \right\}, \quad (21.5)$$

with the “pure” robust stability requirement occupying the upper left diagonal, and the nominal performance requirement on the lower right. Setting  $\alpha = d_2/d_1$  and fixing  $\omega$ , and taking the definition of  $A(\alpha)$  from (21.5), we have

$$\mu(M(j\omega)) = \inf_{|\alpha| > 0} \{ \lambda_{\max}^{1/2}(A^*(\alpha)A(\alpha)) \}. \quad (21.6)$$

Now, for nominal performance, we require that

$$\left| \frac{W_2}{1 + P_0 K}(j\omega) \right| \leq 1. \quad (21.7)$$

For robust stability, we need

$$\left| \frac{W_1 P_0 K}{1 + P_0 K}(j\omega) \right| \leq 1. \quad (21.8)$$

For robust performance, the necessary and sufficient condition is

$$\mu(M(j\omega)) \leq 1. \quad (21.9)$$

A bit of algebra yields

$$\lambda_{\max}(A^*A) = |\alpha|^2 \left| \frac{W_1 K}{1 + P_0 K}(j\omega) \right|^2 + \left| \frac{W_2}{1 + P_0 K}(j\omega) \right|^2 \quad (21.10)$$

$$+ \left| \frac{W_1 K P_0}{1 + P_0 K}(j\omega) \right|^2 + \frac{1}{|\alpha|^2} \left| \frac{W_2 P_0}{1 + P_0 K}(j\omega) \right|^2 \quad (21.11)$$

from which we have

$$\inf_{\alpha} \lambda_{\max}(A^*A) = \left( \left| \frac{W_1 P_0 K}{1 + P_0 K}(j\omega) \right| + \left| \frac{W_2}{1 + P_0 K}(j\omega) \right| \right)^2. \quad (21.12)$$

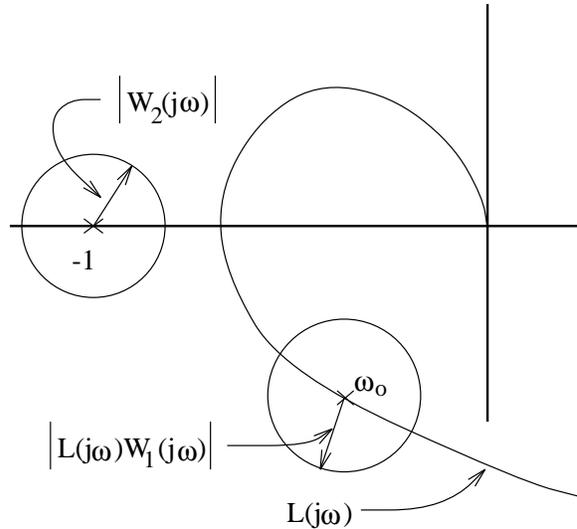


Figure 21.4: Robust Performance/Nyquist Criterion

This minimum occurs at

$$|\alpha|^2 = \frac{|W_2 P_0|}{|W_1 K|} \quad (21.13)$$

which is not equal to 1 in general, so that  $\sup_{\omega} \mu \leq \|M\|_{\infty}$ . In other words,  $\mu$  is a less conservative measure than  $\|\cdot\|_{\infty}$  in this case.

Once again, there is a graphical interpretation of the SISO robust disturbance rejection problem, in terms of the Nyquist criterion. From (21.12), we have

$$\mu(M(j\omega)) \leq 1 \iff \left| \frac{W_1 P_0 K}{1 + P_0 K}(j\omega) \right| + \left| \frac{W_2}{1 + P_0 K}(j\omega) \right| \leq 1. \quad (21.14)$$

Letting  $L(j\omega)$  represent the nominal loop gain  $P_0 K(j\omega)$ , this can be rewritten as:

$$|W_1 L(j\omega)| + |W_2| \leq |1 + L(j\omega)|. \quad (21.15)$$

Graphically, we can represent this at each frequency  $\omega$  as a circle centered at  $-1$  of radius  $|W_2|$ , and a second circle centered at  $L(j\omega)$  of radius  $|W_1 L(j\omega)|$ . Robust performance will be achieved as long as the two circles never intersect.

## Loop-shaping Revisited

Loop-shaping is a well-established method of control design that concentrates on the frequency-domain characteristics of the open-loop transfer function  $L = P_0 K$ . Based primarily on design experience, there are certain characteristics of the loop transfer function that translate into desirable control performance. Other open-loop characteristics are known by experience to result in undesirable or unpredictable behavior. This method differs from  $\mu$ -synthesis and  $\mathcal{H}_{\infty}$  methods, which concentrate on optimizing the characteristics of the closed-loop transfer function. Since, presumably, a controller with good behavior designed by loop-shaping should be similar in some way to a controller designed by more recent methods, it is of interest to look for parallels in the heuristic rules of loop-shaping and the more methodical methods of  $\mu$ -synthesis and  $\mathcal{H}_{\infty}$ .

Identifying the sensitivity and complementary sensitivity functions from (21.14), we can write the RP requirement as

$$|W_1(j\omega)T(j\omega)| + |W_2(j\omega)S(j\omega)| \leq 1. \quad (21.16)$$

Model uncertainty typically increases with frequency, so it is important that the complementary sensitivity function decreases with increasing frequency. For disturbance rejection, which is typically most critical over a low frequency range, we require that  $S(j\omega)$  remain small. The weighting functions  $W_1$  and  $W_2$  are designed to reflect this, and so might take on the form of Figure 21.5. Normally, at low

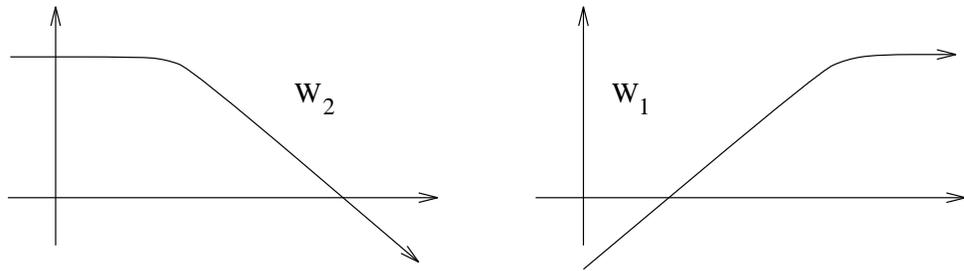


Figure 21.5: Typical Weighting Functions

frequency,  $L(j\omega) \gg 1$  and at high frequency,  $L(j\omega) \ll 1$ . Now,

$$T_0 = \frac{L}{1+L}, \quad S_0 = \frac{1}{1+L} \quad (21.17)$$

so that at low frequency,  $T_0 \approx 1$  and  $S_0 \approx 1/L$ . Thus we can approximate the RP requirement at the low end as:

$$|W_1| + \left| W_2 \frac{1}{L} \right| \leq 1 \quad \implies \quad |L| \geq \frac{|W_2|}{1 - |W_1|} \quad (21.18)$$

At high frequency, the approximation is  $T_0 \approx L$  and  $S_0 \approx 1$ , which leads to:

$$|W_1 L| + |W_2| \leq 1, \quad \implies \quad |L| \leq \frac{1 - |W_2|}{|W_1|}. \quad (21.19)$$

These constraints are summarized in Figure 21.6, which also notes another design rule, which is that the 0 dB crossing should occur at a slope no more negative than -40 dB per decade. If  $W_1$  and  $W_2$  do not overlap significantly in frequency, then the upper and lower bounds reduce to  $|W_2|$  and  $1/|W_1|$ , respectively.

### Example 21.2 (Loop Shaping)

Assume  $P_0$  is minimum phase stable with relative degree 1. Designing a controller by shaping the loop gain  $L = P_0 K$  is not affected by  $P_0$ ; just the relative degree is needed.

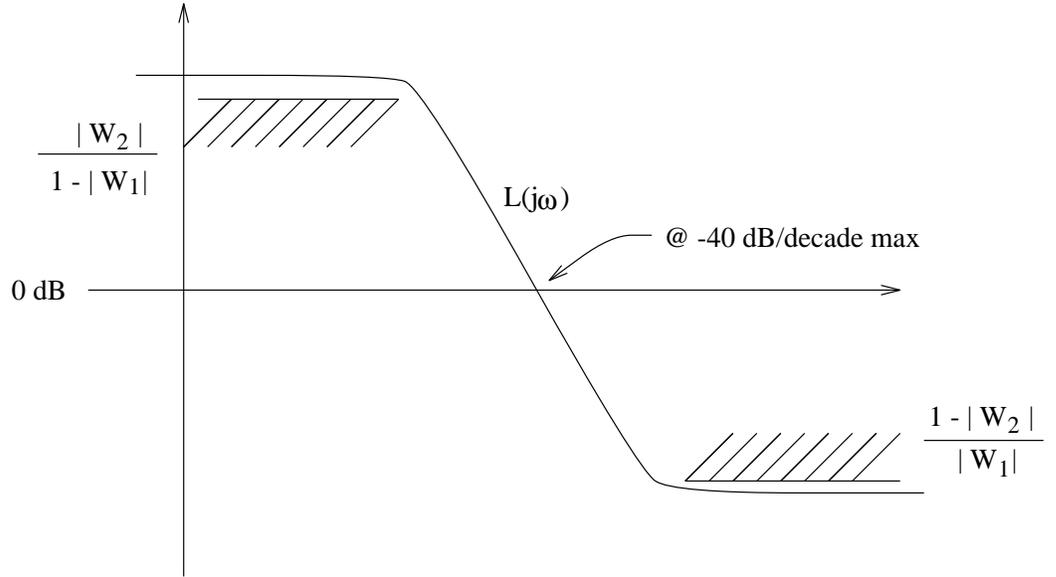


Figure 21.6: Typical Loop-shaping Problem

Suppose the multiplicative uncertainty is described by

$$W_1 = \frac{s + 1}{20(0.01s + 1)},$$

*i.e.*, the multiplicative perturbations of the plant are upper bounded by  $W_1(j\omega)$  at each frequency.

The objective is to track sinusoidal signals at the reference input in the frequency range  $[0, 1]$  rad/s. We would like to make the tracking error small; however, we do not know yet by how much. Let  $W_2(j\omega)$  have the following frequency response

$$|W_2(j\omega)| = \begin{cases} a & 0 \leq \omega \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Note that this may not correspond to a stable  $W_2(s)$ ; however, this does not affect the resulting loop shape. We are going to exhibit the design by trial and error. Let

$$L(s) = \frac{b}{cs + 1}.$$

At high frequency,  $\omega \geq 20$ ,

$$L \leq \frac{1 - |W_2|}{|W_1|} = \frac{1}{|W_1|} \quad \omega \geq 20.$$

If we pick  $c = 1$ , then the largest value for  $b$  such that the above is satisfied is  $b = 20$ . Hence

$$L(s) = \frac{20}{s + 1}.$$

At low frequency,  $\omega \leq 1$ ,

$$|L| \geq \frac{|W_2|}{1 - |W_1|} = \frac{a}{1 - |W_1|}.$$

Since  $|L(j\omega)|$  is decreasing and  $|W_1(j\omega)|$  is increasing in the range  $[0, 1]$ , the largest  $a$  can be solved for:

$$|L(j1)| = \frac{a}{1 - |W_1(j1)|},$$

which implies that  $a = 13.15$ . Checking the RP condition

$$|W_2S(j\omega)| + |W_1T(j\omega)| \leq 0.92 \quad \forall \omega$$

which implies RP is achieved and the tracking error is smaller than  $1/13.15$  in the range  $[0, 1]$ . If a better performance is desired, a possibly more complicated  $L$  needs to be used.

The discussion in this chapter has focused on perturbations that are arbitrary dynamic systems. This allowed us to think of any class of structured perturbations as sets of arbitrary (structured) matrices at each frequency point. These matrices correspond to evaluating the dynamic system at a given frequency.

In practical applications, some perturbations may be static and not dynamic. These arise in problems with real parameter uncertainties. We can still proceed as before and transform such problems to the general  $M$ - $\Delta$  diagram. In this case,  $\Delta$  will have a combination of both static and dynamic perturbations.  $\mu$  for such a class can be defined as before, and it will provide a necessary and sufficient condition for robust stability.

The main issue here is computing a good upper bound for  $\mu$ . Of course, we can always embed this class of perturbations in a larger class containing dynamic perturbations and use  $D$ -scaling to obtain an upper bound. This, however, gives conservative conditions. Computing non-conservative upper bounds of  $\mu$  for such perturbations remains an active area of research.

## 21.7 Rank-One $\mu$

Although we do not have methods for computing  $\mu$  exactly, there is one particular situation where this is possible. This situation occurs if  $M$  has rank 1, *i.e.*

$$M = ab^*$$

where  $a, b \in \mathbb{C}^n$ . Then it follows that  $\mu$  with respect to  $\Delta$  containing complex diagonal perturbations is given by

$$\frac{1}{\mu(M)} = \inf_{\Delta \in \Delta} \{\sigma_{\max}(\Delta) \mid \det(I - M\Delta) = 0\}.$$

However,

$$\begin{aligned} \det(I - M\Delta) &= \det(I - ab^*\Delta) \\ &= \det(I - b^*\Delta a) \\ &= \det \left( I - [\Delta_1 \cdots \Delta_n] \begin{bmatrix} \bar{b}_1 a_1 \\ \bar{b}_2 a_2 \\ \vdots \\ \bar{b}_n a_n \end{bmatrix} \right) \\ &= 1 - [\Delta_1 \cdots \Delta_n] \begin{bmatrix} \bar{b}_1 a_1 \\ \bar{b}_2 a_2 \\ \vdots \\ \bar{b}_n a_n \end{bmatrix}, \end{aligned}$$

and  $\sigma_{\max}(\Delta) = \max_i |\Delta_i|$ . Hence,

$$\frac{1}{\mu(M)} = \inf_{\Delta_1, \dots, \Delta_n} \left\{ \max_i |\Delta_i| \left| [\Delta_1 \cdots \Delta_n] \begin{bmatrix} \bar{b}_1 a_1 \\ \bar{b}_2 a_2 \\ \vdots \\ \bar{b}_n a_n \end{bmatrix} = 1 \right. \right\}.$$

Optimizing the RHS, it follows that (verify)

$$\frac{1}{\mu(M)} = \frac{1}{\sum_{i=1}^n |\bar{b}_i a_i|} \leftrightarrow \mu(M) = \sum_{i=1}^n |\bar{b}_i a_i|.$$

Notice that the SISO robust disturbance rejection problem is a rank-one problem. This follows since

$$M = \begin{bmatrix} -W_1 K \\ W_2 \end{bmatrix} \begin{bmatrix} \frac{P_0}{1 + P_0 K} & \frac{1}{1 + P_0 K} \end{bmatrix}.$$

Then

$$\mu(M(j\omega)) = \left| \frac{W_1 P_0 K}{1 + P_0 K}(j\omega) \right| + \left| \frac{W_2}{1 + P_0 K}(j\omega) \right|$$

which is the condition we derived before.

### Coprime Factor Perturbations

Consider the class of SISO systems

$$\Omega = \left\{ \frac{N(s)}{D(s)} \mid N = N_0 + \Delta_1 W_1, D = D_0 + \Delta_2 W_2, \|\Delta_i\| < 1 \right\}$$

where the nominal plant is  $N_0/D_0$  with the property that both  $N_0$  and  $D_0$  are stable with no common zeros in the RHP. Assume that  $K$  stabilizes  $N_0/D_0$ . This block diagram is shown in Figure 21.7.

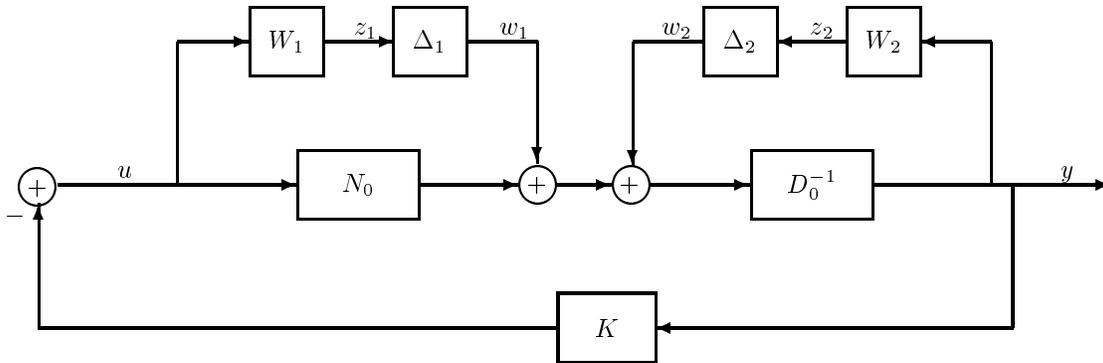


Figure 21.7: Coprime Factor Perturbation Model

The closed loop block diagram can be mapped to the  $M$ - $\Delta$  diagram where

$$\begin{aligned} M &= \begin{bmatrix} -\frac{W_1 K}{D_0 + N_0 K} & -\frac{W_1 K}{D_0 + N_0 K} \\ \frac{W_2}{D_0 + N_0 K} & \frac{W_2}{D_0 + N_0 K} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{W_1 K}{D_0 + N_0 K} \\ \frac{W_2}{D_0 + N_0 K} \end{bmatrix} [1 \quad 1]. \end{aligned}$$

Hence,  $M$  has rank 1 and

$$\mu(M(j\omega)) = \left| \frac{W_1 K}{D_0 + N_0 K} \right| + \left| \frac{W_2}{D_0 + N_0 K} \right|.$$

### Robust Hurwitz Stability of Polynomials with Complex Perturbations

Another application of the structured singular value with rank one matrices is the robust stability of a family of polynomials with complex perturbations of the coefficients. In this case let  $\delta = [\delta_{n-1} \quad \delta_{n-2} \quad \dots \quad \delta_0]^T$  and consider the polynomial family

$$P(s, \delta) = s^n + (a_{n-1} + \gamma_{n-1}\delta_{n-1})s^{n-1} + \dots + (a_0 + \gamma_0\delta_0),$$

where  $a_i$ ,  $\gamma_i$ , and  $\delta_i \in \mathbb{C}$  and  $|\delta_i| \leq 1$ . We want to obtain a condition that is both necessary and sufficient for the Hurwitz stability of the entire family of polynomials  $P(s, \delta)$ . We can write the polynomials in this family as

$$P(s, \delta) = P(s, 0) + \tilde{P}(s, \delta) \tag{21.20}$$

$$= (s^n + a_{n-1}s^{n-1} + \dots + a_0) + (\gamma_{n-1}\delta_{n-1}s^{n-1} + \dots + \gamma_0\delta_0), \tag{21.21}$$

which can also be rewritten as

$$P(s, \delta) = P(s, 0) + [1 \quad 1 \quad \dots \quad 1] \begin{bmatrix} \delta_{n-1} & 0 & 0 & \dots & 0 \\ 0 & \delta_{n-2} & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & \delta_1 & 0 \\ & & & 0 & \delta_0 \end{bmatrix} \begin{bmatrix} \gamma_{n-1}s^{n-1} \\ \gamma_{n-2}s^{n-2} \\ \vdots \\ \gamma_1 s \\ \gamma_0 \end{bmatrix}.$$

We assume that the center polynomial  $P(s, 0)$  is Hurwitz stable. This implies that the stability of the entire family  $P(s, \delta)$  is equivalent to the condition that

$$1 + \frac{1}{P(j\omega, 0)} [1 \quad 1 \quad \dots \quad 1] \begin{bmatrix} \delta_{n-1} & 0 & 0 & \dots & 0 \\ 0 & \delta_{n-2} & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & \delta_1 & 0 \\ & & & 0 & \delta_0 \end{bmatrix} \begin{bmatrix} \gamma_{n-1}(j\omega)^{n-1} \\ \gamma_{n-2}(j\omega)^{n-2} \\ \vdots \\ \gamma_1(j\omega) \\ \gamma_0 \end{bmatrix} \neq 0$$

for all  $\omega \in \mathbb{R}$  and  $|\delta_i| \leq 1$ . This is equivalent to the condition that

$$\det \left( I + \frac{1}{P(j\omega, 0)} \begin{bmatrix} \gamma_{n-1}(j\omega)^{n-1} \\ \gamma_{n-2}(j\omega)^{n-2} \\ \vdots \\ \gamma_1(j\omega) \\ \gamma_0 \end{bmatrix} [ 1 \ 1 \ \dots 1 ] \Delta \right) \neq 0$$

for all  $\omega \in \mathbb{R}$  and  $\Delta \in \Delta$  with  $\|\Delta\|_\infty \leq 1$ . Now using the concept of the structured singular value we arrive at the following condition which is both necessary and sufficient for the Hurwitz stability of the entire family

$$\mu(M(j\omega)) < 1$$

for all  $\omega \in \mathbb{R}$ , where

$$M(j\omega) = \frac{1}{P(j\omega, 0)} \begin{bmatrix} \gamma_{n-1}(j\omega)^{n-1} \\ \gamma_{n-2}(j\omega)^{n-2} \\ \vdots \\ \gamma_1(j\omega) \\ \gamma_0 \end{bmatrix} [ 1 \ 1 \ \dots 1 ] .$$

Clearly this is a rank one matrix and by our previous discussion the structured singular value can be computed analytically resulting in the following test

$$\frac{1}{|P(j\omega, 0)|} \sum_{i=1}^n |\gamma_{n-i}| |\omega|^{n-i} < 1$$

for all  $\omega \in \mathbb{R}$ .

## Exercises

**Exercise 21.1** In decentralized control, the plant is assumed to be diagonal and controllers are designed independently for each diagonal element. If however, the real process is not completely decoupled, the interactions between these separate subsystems can drive the system to instability.

Consider the  $2 \times 2$  plant

$$P(s) = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}.$$

Assume that  $P_{12}$  and  $P_{21}$  are stable and relatively small in comparison to the diagonal elements, and only a bound on their frequency response is available. Suppose a controller  $K = \text{diag}(K_1, K_2)$  is designed to stabilize the system  $P_0 = \text{diag}(P_{11}, P_{22})$ .

1. Set-up the problem as a stability robustness problem, i.e., put the problem in the  $M - \Delta$  form.
2. Derive a non-conservative condition (necessary and sufficient) that guarantees the stability robustness of the above system. Assume the off-diagonal elements are perturbed independently. Reduce the result to the simplest form (an answer like  $\mu(M) < 1$  is not acceptable; this problem has an exact solution which is computable).
3. How does your answer change if the off-diagonal elements are perturbed simultaneously with the same  $\Delta$ .

**Exercise 21.2** Consider the rank 1  $\mu$  problem. Suppose  $\Delta$ , contains only real perturbations. Compute the exact expression of  $\mu(M)$ .

**Exercise 21.3** Consider the set of plants characterized by the following sets of numerators and denominators of the transfer function:

$$N(s) = N_0(s) + N_\delta(s)\delta, \quad D(s) = D_0(s) + D_\delta(s)\delta$$

Where both  $N_0$  and  $D_0$  are polynomials in  $s$ ,  $\delta \in \mathbb{R}^n$ , and  $N_\delta$ ,  $D_\delta$  are polynomial row vectors. The set of all plants is then given by:

$$\Omega = \left\{ \frac{N(s)}{D(s)} \mid \delta \in \mathbb{R}^n, |\delta_i| \leq \gamma \right\}$$

Let  $K$  be a controller that stabilizes  $\frac{N_0}{D_0}$ . Compute the exact stability margin; i.e., compute the largest  $\gamma$  such that the system is stable.

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