Lectures on Dynamic Systems and Control

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Chapter 15

External Input-Output Stability

15.1 Introduction

In this lecture, we introduce the notion of external, or input-output, stability for systems. There are many connections between this notion of stability and that of Lyapunov stability which we discussed in the previous two chapters. We will only make the connection in the LTI case. In addition, we will point out the fact that the notion of input-output stability depends in a non-trivial fashion on the way we measure the inputs and the outputs.

15.2 Signal Measures

The signals of interest to us are defined as maps from a time set into $\mathbb{R}^n$. A continuous-time signal is a map from $\mathbb{R} \to \mathbb{R}^n$, and a discrete-time signal is a map from $\mathbb{Z} \to \mathbb{R}^n$. If $n = 1$ we have a scalar signal, otherwise we have a vector-valued signal. It is helpful, in understanding the various signal measures defined below, to visualize a discrete-time signal $w(k)$ as just a *vector* of infinite (or, if our signal is defined only for non-negative time, then a vector of semi-infinite) length or dimension, concretely representing it as the array

$$\begin{pmatrix} \vdots \\ w(0) \\ w(1) \\ \vdots \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} w(0) \\ w(1) \\ \vdots \end{pmatrix} . \tag{15.1}$$

Three of the most commonly used DT signal measures are then natural generalizations of the finite-dimensional vector norms (∞-, 2- and 1-norms) that we have already encountered in earlier chapters, generalized to such infinite-dimensional vectors. We shall examine these three measures, and a fourth that is related to the 2-norm, but is not quite a norm. We shall also define CT signal measures that are natural counterparts of the DT measures.

The signal measures that we study below are:

1. peak magnitude (or ∞-norm);
2. energy (whose square root is the 2-norm);
3. power (or mean energy, whose square root is the "rms" or root-mean-square value);
4. "action" (or 1-norm).

Peak Magnitude: The ∞-Norm

The ∞-norm $\|w\|_\infty$ of a signal is its peak magnitude, evaluated over all signal components and all times:

$$\begin{aligned}
\|w\|_\infty &\stackrel{\triangle}{=} \text{max magnitude of } w \\
&\stackrel{\triangle}{=} \sup_k \max_i |w_i(k)| \;=\; \sup_k \|w(k)\|_\infty \qquad \text{(for DT systems)} \qquad (15.2)\\
&\stackrel{\triangle}{=} \sup_t \max_i |w_i(t)| \;=\; \sup_t \|w(t)\|_\infty \qquad \text{(for CT systems)} \quad , \qquad (15.3)
\end{aligned}$$

where $w_i(k)$ indicates the i-th component of the signal vector $w(k)$. Note that $\|w(k)\|_\infty$ denotes the ∞-norm of the signal value *at time* k, i.e. the familiar ∞ norm of an n-vector, namely the maximum magnitude among its components. On the other hand, the notation $\|w\|_\infty$ denotes the ∞-norm of the *entire signal*. The "sup" denotes the *supremum* or *least upper bound*, the value that is approached arbitrarily closely but never (i.e., at any finite time) exceeded. We use "sup" instead of "max" because over an infinite time set the signal magnitude may not have a maximum, i.e. a peak value that is actually attained — consider, for instance, the simple case of the signal

$$1 - \frac{1}{1+|k|} \, ,$$

which does not attain its supremum value of 1 for any finite k.

Note that the DT definition is the natural generalization of the standard ∞-norm for finite-dimensional vectors to the case of our infinite vector in (15.1), while the CT definition is the natural counterpart of the DT definition. This pattern is typical for all the signal norms we deal with, and we shall not comment on it explicitly again.

Example 15.1 Some bounded signals:

(a) For $w(t) = 1$, $t \in \mathbb{R}$, $t \geq 0$:
$\|w\|_\infty = 1$.

(b) For $w(t) = a^t$, $t \in \mathbb{Z}$:
$\|w\|_\infty = \infty$ if $|a| \neq 1$ and $\|w\|_\infty = 1$ otherwise.

The space of all signals with finite ∞-norm are generally denoted by ℓ_∞ and $\mathcal{L}_\infty$ for DT and CT signals respectively. For vector-valued signals, the size of the vector may be explicitly added to the symbol, e.g., ℓ_∞^n. These form normed-vector spaces.

Energy and the 2-Norm

The 2-norm of a signal is the square root of its "energy", which is in turn defined as the sum (in DT) or integral (in CT) of the squares of all components over the entire time set:

$$\begin{aligned} \|w\|_2 &\triangleq \text{square-root of } \textit{energy} \text{ in } w \\ &\triangleq \left[\sum_k w^T(k)w(k)\right]^{\frac{1}{2}} = \left[\sum_k \|w(k)\|_2^2\right]^{\frac{1}{2}} \qquad \text{(for DT systems)} \qquad (15.4) \\ &\triangleq \left[\int w^T(t)w(t)\,dt\right]^{\frac{1}{2}} = \left[\int_t \|w(t)\|_2^2\,dt\right]^{\frac{1}{2}} \qquad \text{(for CT systems)} \quad . \quad (15.5) \end{aligned}$$

Example 15.2 Some examples:

(a) For $w(t) = e^{-at}$ and time set $t \geq 0$, with $a > 0$:
$\|w\|_2 = \frac{1}{\sqrt{2a}} < \infty$

(b) For $w(t) = 1$ and time set $t \geq 0$:
$\|w\|_2 = \infty$

(c) For $w(t) = \cos\omega_o t$ and time set $t \geq 0$:
$\|w\|_2 = \infty$.

These examples suggest that bounded-energy signals go to zero as time progresses. For discrete-time signals, this expectation holds up: if $\|w\|_2 < \infty$, then $\|w(k)\| \longrightarrow 0$ as $k \longrightarrow \infty$. However, for continuous-time signals, the property of having bounded energy does not imply that $\|w(t)\| \longrightarrow 0$ as $t \longrightarrow \infty$, unless additional assumptions are made. This is because continuous-time bounded energy signals can still have arbitrarily large excursions in amplitude, provided these excursions occur over sufficiently narrow intervals of time that the integral of the square remains finite — consider, for instance, a CT signal that is zero everywhere, except for a triangular pulse of height k and base $1/k^4$ centered at every nonzero integer value k. If the continuous-time signal $w(t)$ is differentiable and both w and its derivative $\dot{w}$ have bounded energy (which is *not* the case for the preceding triangular-pulse example), then it *is* true that $\|w(t)\| \longrightarrow 0$ as $t \longrightarrow \infty$. The reader may wish to verify this fact.

It is not hard to show that DT or CT signals with finite 2-norms form a vector space. On the vector space ℓ_2 (respectively $\mathcal{L}_2$) of DT (respectively CT) signals with finite 2-norm, one can define a natural inner product as follows, between signals x and y:

$$\langle x\,,\,y\rangle \triangleq \left[\sum_k x^T(k)y(k)\right] \qquad \text{(for DT systems)} \qquad (15.6)$$

$$\triangleq \left[\int x^T(t)y(t)\,dt\right] \qquad \text{(for CT systems)} \quad . \tag{15.7}$$

(The 2-norm is then just the square root of the inner product of a signal with itself.) These particular infinite-dimensional inner-product vector spaces are of great importance in applications, and are the prime examples of what are known as Hilbert spaces.

Power and RMS Value

Another signal measure of interest is the "power" or mean energy of the signal. One also often deals with the square root of the power, which is commonly termed the "root-mean-square" (or "rms") value. For a signal w for which the following limits exist, we define the power by

$$P_w \triangleq \lim_{N\to\infty}\left[\frac{1}{2N}\sum_{k=-(N-1)}^{N-1} w^T(k)w(k)\right] \qquad \text{(for discrete} - \text{time systems)} \tag{15.8}$$

$$\triangleq \lim_{L\to\infty}\left[\frac{1}{2L}\int_{-L}^{L} w^T(t)w(t)dt\right] \qquad \text{(for continuous} - \text{time systems)} \quad . \tag{15.9}$$

(The above definitions assume that the time set is the entire time axis, but the necessary modifications for other choices of time set should be obvious.) We shall use the symbol ρ_w to denote the rms value, namely $\sqrt{P_w}$. The reason that ρ_w is *not* a norm, according to the technical definition of a norm, is that $\rho_w = 0$ does *not* imply that $w = 0$.

Example 15.3 Some finite-power signals:

(a) For $w(t) = 1$:
$\rho_w = 1$

(b) For $w(t)$ such that $\|w\|_2 < \infty$:
$\rho_w = 0$

(c) For $w(t) = \cos\omega_0 t$ (with $t \in \mathbb{R}$ or $t \in \mathbb{Z}$):
$\rho_w = \frac{1}{\sqrt{2}}$.

Example c) points out an important difference between bounded power and bounded energy signals: unlike bounded energy signals, if $\rho_w < \infty$, the signal doesn't necessarily decay to zero.

As a final comment on the definition of the power of a signal, we elaborate on the hint in the preamble to our definition that the limit required by the definition may not exist for certain signals. The limit of a sequence or function (in our case, the sequence or function is the set of finite-interval rms values, considered over intervals of increasing length) may not exist even if the sequence or function stays bounded, as when it oscillates between two different finite values. The following signal is an example of a CT signal that is bounded but does not have a well-defined power, because the required limit does not exist:

$$w(t) = \begin{cases} 1 & \text{if } t \in [2^{2k}, 2^{2k+1}], \text{ for } k = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Also note that the desired limit may exist, but not be finite. For instance, the limit of a sequence is $+\infty$ if the values of the sequence remain above any chosen finite positive number for sufficiently large values of the index.

Action: The 1-Norm

The 1-norm of a signal is also sometimes termed the "action" of the signal, which is in turn defined as the sum (in DT) or integral (in CT) of the 1-norm of the signal value at each time, taken over the entire time set:

$$\begin{aligned} \|w\|_1 &\triangleq \textit{action} \text{ of } w \\ &\triangleq \left[\sum_k \|w(k)\|_1\right] \qquad \text{(for discrete} - \text{time systems)} \qquad (15.10) \\ &\triangleq \left[\int \|w(t)\|_1 \, dt\right] \qquad \text{(for continuous} - \text{time systems)} \quad . \qquad (15.11) \end{aligned}$$

Recall that $\|w(k)\|$ for the n-vector $w(k)$ denotes the sum of magnitudes of its components.

The space of all signals with finite 1-norm are generally denoted by ℓ_1 and $\mathcal{L}_1$ for DT and CT signals respectively. These form normed-vector spaces.

We leave you to construct examples that show familiar signals of finite and infinite 1-norm.

Relationships Among Signal Measures

a) If w is a discrete-time sequence, then

$$\|w\|_2 < \infty \Longrightarrow \|w\|_\infty < \infty \qquad (15.12)$$

but

$$\|w\|_2 < \infty \not\Longleftarrow \|w\|_\infty < \infty \qquad (15.13)$$

b) If w is a continuous-time signal, then

$$\|w\|_2 < \infty \not\Longrightarrow \|w\|_\infty < \infty. \qquad (15.14)$$

and

$$\|w\|_2 < \infty \not\Longleftarrow \|w\|_\infty < \infty. \qquad (15.15)$$

c) If $\|w\|_\infty < \infty$, then (when ρ_w exists)

$$\rho_w \leq \|w\|_\infty$$

Item a) is true because of the relationship between energy and magnitude for discrete-time signals. Since the energy of a DT signal is the sum of squared magnitudes, if the energy is bounded, then the magnitude must be bounded. However, the converse is not true —take for example, the signal $w(k) = 1$. As item b) indicates, though, bounded energy implies nothing about the boundedness of magnitude for continuous time signals.

(Many more relationships of the above form can be stated.)

15.3 Input-Output Stability

At this point, it is important to make a connection between the stability of a system and its input-output behavior. The most important notion is that of ℓ_p-stability (p-stability).

Definition 15.1 A system with input signal u and output signal y that is obtained from u through the action of an arbitrary operator H, so $y = H(u)$, is ℓ_p-stable or p-stable ($p = 1,\ 2,\ \infty$) if there exists a finite $C \in \mathbb{R}$ such that

$$\|y\|_p \leq C\|u\|_p \tag{15.16}$$

for every input u.

A p-stable system is therefore characterized by the requirement that every input of finite p-norm gives rise to an output of finite p-norm. For the case $p = \infty$, this notion is known as Bounded-Input Bounded-Output (BIBO) stability. We will see that BIBO stability is equivalent to p-stability for finite-dimensional LTI state-space systems, but not necessarily in other cases.

Example 15.4 The system described by one integrator:

$$\dot{y} = u$$

is not BIBO stable. A step input is mapped to a ramp which is unbounded. It is not hard to see that this system is not p-stable for any p.

15.3.1 BIBO Stability of LTI Systems

A continuous-time LTI system may be characterized by its impulse response *matrix*, $\mathcal{H}(\,\cdot\,)$, whose (i,j)th entry $h_{ij}(\,\cdot\,)$ is the impulse response from the jth input to the ith output. In other words the input-output relation is given by

$$y(t) = \int \mathcal{H}(t-\tau)u(\tau)d\tau\ .$$

Theorem 15.1 A CT LTI system with m inputs, p outputs, and impulse response matrix $\mathcal{H}(t)$ is BIBO stable if and only if

$$\max_{1\leq i\leq p} \sum_{j=1}^{m} \int |h_{ij}(t)|\, dt < \infty.$$

Proof: The proof of sufficiency involves a straightforward computation of bounds. If u is an input signal that satisfies $\|u\|_\infty < \infty$, i.e. a bounded signal, then we have

$$y(t) = \int \mathcal{H}(t-\tau)u(\tau)d\tau,$$

and

$$\begin{aligned} \max_{1\le i\le p} |y_i(t)| &= \max_i \left| \int \sum_{j=1}^{m} h_{ij}(t-\tau)u_j(\tau)\, d\tau \right| \\ &\le \left[\max_i \int \sum_j |h_{ij}(t-\tau)|\, d\tau \right] \max_j \sup_t |u_j(t)|. \end{aligned}$$

It follows that

$$\|y\|_\infty = \sup_t \max_i |y_i(t)| \le \left[\max_i \sum_j \int |h_{ij}(t)| dt \right] \|u\|_\infty < \infty.$$

In order to prove the converse of the theorem, we show that if the above integral is infinite then there exists a bounded input that will be mapped to an unbounded output. Let us consider the case when $p = m = 1$, for notational simplicity (in the general case, we can still narrow the focus to a single entry of the impulse response matrix). Denote the impulse response by $h(t)$ for this scalar case. If the integral

$$\int |h(t)|\, dt$$

is unbounded then given any (large) M there exists an interval of length $2T$ such that

$$\int_{-T}^{T} |h(t)|\, dt > M.$$

Now by taking the input $u_M(t)$ as

$$u_M(t) = \begin{cases} sgn(h(-t)) & -T \le t \le T \\ 0 & |t| > T \end{cases},$$

we obtain an output $y_M(t)$ that satisfies

$$\begin{aligned} \sup_t |y_M(t)| \ge y_M(0) &= \int_{-T}^{T} h(0-\tau)u_M(\tau)\, d\tau \\ &= \int_{-T}^{T} |h(0-\tau)|\, d\tau \\ &> M. \end{aligned}$$

In other words, for any $M > 0$, we can have an input whose maximum magnitude is 1 and whose corresponding output is larger than M. Therefore, there is no finite constant C such that the inequality (24.3) holds.

Further reflection on the proof of Theorem 15.1 reveals that the constant $\|\mathcal{H}\|_1$ defined by

$$\|\mathcal{H}\|_1 = \max_i \sum_j \int |h_{ij}(t)| dt$$

is the smallest constant C that satisfies the inequalty (24.3) when $p = \infty$. This number is called the ℓ_1-norm of $\mathcal{H}(t)$. In the scalar case, this number is just the ℓ_1-norm of $h(\cdot)$, regarded as a signal.

The discrete-time case is quite similar to continuous-time where we start with a pulse response *matrix*, $\mathcal{H}(\cdot)$, whose (i,j)th entry $h_{ij}(\cdot)$ is the pulse response from the jth input to the ith output. The input-output relation is given by

$$y(t) = \sum_\tau \mathcal{H}(t-\tau)u(\tau) \ .$$

Theorem 15.2 A DT LTI system with m inputs, p outputs, and pulse response matrix $\mathcal{H}(t)$ is BIBO stable if and only if

$$\max_{1 \le i \le p} \sum_{j=1}^{m} \sum_t |h_{ij}(t)| < \infty.$$

In addition, the constant $\|\mathcal{H}\|_1$ defined by

$$\|\mathcal{H}\|_1 = \max_i \sum_j \sum_t |h_{ij}(t)|$$

is the smallest constant C that satisfies the inequalty (24.3) when $p = \infty$. We leave the proof of these facts to the reader.

Application to finite-dimensional State-Space Models

Now consider the application to the following causal CT LTI system in state-space form (and hence of finite order):

$$\dot{x} = Ax + Bu \qquad (15.17)$$

$$y = Cx + Du \qquad (15.18)$$

The impulse response of this system is given by

$$\mathcal{H}(t) = Ce^{At}B + D\delta(t) \ \ \text{for } t \ge 0$$

which has Laplace transform

$$H(s) = C(sI - A)^{-1}B + D$$

The system (15.18) is BIBO stable if and only if the poles of $H(s)$ are in the open left half plane. (We leave the proof to you.) This is in turn guaranteed if the system is asymptotically stable, i.e. if A has all its eigenvalues in the open left half plane.

Example 15.5 BIBO Stability Doesn't Imply Asymptotic Stability

It is possible that a system be BIBO stable and not asymptotically stable. Consider the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & -1 \end{pmatrix} x \end{aligned}$$

This system is not stable since A has an eigenvalue at 1. Nevertheless, thanks to a pole-zero cancellation, the only pole that $H(s)$ has is at -1, so the system is BIBO stable. We shall have much more to say about such cancellations in the context of reachability, observability, and minimality (the example here turns out to be unobservable).

Marginal stability of an LTI system, i.e., stability in the sense of Lyapunov but without asymptotic stability, is not sufficient to guarantee BIBO stability. For instance, consider a simple integrator, whose transfer function is $1/s$.

Time-Varying and Nonlinear Systems

Although there are results connecting Lyapunov stability with I/O stability for general time-varying and nonlinear systems, they are not as powerful as the linear time-invariant case. In particular, systems may be I/O stable with respect to one norm and not stable with respect to another. Below are some examples illustrating these facts.

Example 15.6 A Time-Varying System

Consider the time-varying DT system given by:

$$y(t) = H(u)(t) = u(0).$$

H is obviously ∞-stable with gain less than 1. However, it is not 2-stable.

Example 15.7 A Nonlinear System

Consider the nonlinear system given by:

$$\dot{x} = -x + e^x u, \qquad y = x.$$

The unforced system is linear and is asymptotically stable. On the other hand the system is not I/O stable. To see this, consider the input $u(t) = 1$. Since $e^x > x$, $\dot{x}$ is always strictly positive, indicating that x is strictly increasing. Hence, for a bounded input, the output is not bounded.

15.3.2 p-Stability of LTI Systems (optional)

In this section we will continue our analysis of the p-stability of systems described through input-output relations. Let us start with the continuous-time case, and restrict ourselves to single-input single-output. The input $u(t)$ is related to the output $y(t)$ by

$$y(t) = \int h(t-\tau)u(\tau)d\tau$$

where $h(t)$ is the impulse response. The following theorem shows that the constant C in 24.3 is always bounded above by $\|h\|_1$.

Theorem 15.3 If $\|h\|_1 < \infty$ and $\|u\|_p < \infty$ then $\|y\|_p < \infty$ and furthermore

$$\|y\|_p \leq \|h\|_1 \|u\|_p \ .$$

Proof: In Theorem 15.1 we have already established this result for $p = \infty$. In what follows $p = 1, 2$. The output $y(t)$ satisfies

$$|y(t)|^p = |(h * u)(t)|^p = \left| \int_{-\infty}^{\infty} h(t-\tau)u(\tau)\ d\tau \right|^p \leq \left(\int_{-\infty}^{\infty} |h(t-\tau)|\ |u(\tau)|\ d\tau \right)^p$$

therefore,

$$\|h * u\|_p^p = \int_{-\infty}^{\infty} |(h * u)(t)|^p\ dt \leq \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |h(t-\tau)|\ |u(\tau)|\ d\tau \right)^p\ dt\ .$$

Next we analyze the inner integral

$$\begin{aligned}
\int_{-\infty}^{\infty} |h(t-\tau)|\ |u(\tau)|\ d\tau &= \int_{-\infty}^{\infty} |h(t-\tau)|^{1/q}\ |h(t-\tau)|^{1/p}\ |u(\tau)|\ d\tau \\
&\leq \left(\int_{-\infty}^{\infty} |h(t-\tau)|\ d\tau \right)^{1/q} \left(\int_{-\infty}^{\infty} |h(t-\tau)|\ |u(\tau)|^p\ d\tau \right)^{1/p}
\end{aligned}$$

where the last inequality follows from Minkowski's inequalities, and $\frac{1}{p} + \frac{1}{q} = 1$. Hence,

$$\begin{aligned}
\|h * u\|_p^p &\leq \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |h(t-\tau)|\ d\tau \right)^{p/q} \left(\int_{-\infty}^{\infty} |h(t-\tau)|\ |u(\tau)|^p\ d\tau \right)\ dt \\
&= \int_{-\infty}^{\infty} (\|h\|_1)^{p/q} \left(\int_{-\infty}^{\infty} |h(t-\tau)|\ |u(\tau)|^p\ d\tau \right)\ dt \\
&= \|h\|_1^{p/q} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |h(t-\tau)|\ |u(\tau)|^p\ d\tau\ dt \\
&= \|h\|_1^{p/q} \int_{-\infty}^{\infty} |u(\tau)|^p \left(\int_{-\infty}^{\infty} |h(t-\tau)|\ dt \right)\ d\tau \\
&= \|h\|_1^{p/q+1} \int_{-\infty}^{\infty} |u(\tau)|^p\ d\tau \\
&= \|h\|_1^p\ \|u\|_p^p
\end{aligned}$$

Therefore

$$\|h * u\|_p \leq \|h\|_1 \; \|u\|_p \; .$$

Recall that when $p = \infty$, $\|h\|_1$ was the smallest constant for which the inequality $\|y\|_p \leq C\|u\|_p$ for all u. This is not the case for $p = 2$, and we will see later that a smaller constant can be found. We will elaborate on these issues when we discuss systems' norms later on in the course. The discrete-time case follows in exactly the same fashion.

Example 15.8 For a finite-dimensional state-space model, a system H is p-stable if and only if all the poles of of $H(s)$ are in the LHP. This coincides with BIBO stability.

Exercises

Exercise 15.1 Non-causal Systems In this chapter, we only focused on causal operators, although the results derived were more general. As an example, consider a particular CT LTI system with a bi-lateral Laplace transform:

$$G(s) = \frac{s+2}{(s-2)(s+1)} .$$

(a) Check the *p-stability* and *causality* of the system in the following cases:

(i) the ROC (Region of Convergence) is $R_1 = \{s \in \mathbb{C} | \ Re(s) < -1\}$ where $Re(s)$ denotes the real part of s;

(ii)the ROC is $R_2 = \{s \in \mathbb{C} | \ -1 < Re(s) < 2\}$;

(iii) the ROC is $R_3 = \{s \in \mathbb{C} | \ Re(s) > 2\}$.

(b) In the cases where the system is not *p-stable* for $p = 2$ and $p = \infty$, find a bounded input that makes the output unbounded, i.e., find an input $u \in L_p$ that produces an output $y \notin L_p$, for $p = 2, \ \infty$.

Exercise 15.2 In nonlinear systems, p-stability may be satisfied in only a local region around zero. In that case, a system will be locally p-stable if:

$$\|Gu\|_p \leq C\|u\|_p, \qquad \text{for all } u \text{ with } \|u\|_p \leq \delta$$

Consider the system:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ z &= Cx + Du \\ y &= g(y) \end{aligned}$$

Where g is a continuous function on $[-T, T]$. Which of the following systems is p-stable, locally p-stable or unstable for $p \geq 1$:

(a) $g(x) = \cos x$.

(b) $g(x) = \sin x$.

(c) $g(x) = \text{Sat}(x)$ where

$$\text{Sat}(x) = \begin{cases} x & |x| \leq 1 \\ 1 & |x| \geq 1 \end{cases}$$

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Spring 2011

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