Lectures on Dynamic Systems and Control

Mohammed Dahleh   Munther A. Dahleh   George Verghese
Department of Electrical Engineering and Computer Science
Massachuasetts Institute of Technology[1]

Chapter 6

Dynamic Models

6.1 Introduction: Signals, Systems and Models

A **system** may be thought of as something that imposes *constraints* on — or enforces relationships among — a set of variables. This "system as constraints" point of view is very general and powerful. Rather more restricted, but still very useful and common, is the view of a system as a *mapping* from a set of *input* variables to a set of *output* variables; a mapping is evidently a very particular form of constraint.

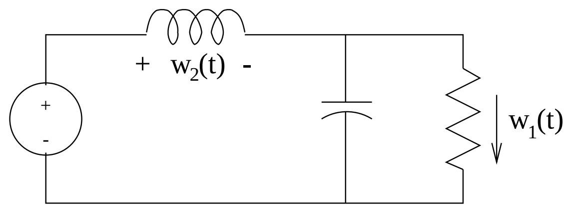
A **(behavioral) model** lists the variables of interest (the *"manifest"* variables) and the constraints that they must satisfy. Any combination of variables that satisfies the constraints is possible or allowed, and is termed *a behavior* of the model.

To facilitate the specification of the constraints, one may introduce auxiliary (*"latent"*) variables. One might then distinguish among the manifest behavior, latent behavior, and *full* behavior (manifest as well as latent).

For a **dynamic model**, the "variables" referred to above are actually *signals* that evolve as a function of time (and/or a function of other independent variables, e.g. space). We first need to specify a *time axis* $\mathbb{T}$ (discrete, continuous, infinite, semi-infinite …) and a *signal space* $\mathbb{W}$, *i.e.* the space of values the signals live in at each time instant. A dynamic model for a set of signals $\{w_i(t)\}$ is then completed by listing the constraints that the $w_i(t)$ must satisfy. Any combination $w(t) = [\ w_1(t), \ \cdots, \ w_\ell(t)\ ]$ of signals that satisfies the constraints is *a behavior* of the model, $w(t) \in \mathbb{B}$, where $\mathbb{B}$ denotes *the behavior.*

We now present some examples of dynamic models, to highlight various possible model representations.

Example 6.1 (Circuit)



Suppose the signals (variables) of interest — the manifest signals — in the above circuit diagram are $w_1(t)$, $w_2(t)$ and $w_3(t)$ for $t \geq 0$, so the signal space $\mathbb{W}$ is $\mathbb{R}^3$ and the time axis $\mathbb{T}$ is $\mathbb{R}^+$ (i.e. the interval $[0, \infty]$). Picking all other component voltages and currents as latent signals, we can write the constraints that define the model as:

$$\begin{cases} \text{2 Kirchhoff's voltage law (KVL) equations} \\ \text{2 Kirchhoff's current law (KCL) equations} \\ \text{4 defining equations for the components} \end{cases}$$

Any set of manifest and latent signals that simultaneously satisfies (or solves) the preceding constraint equations constitutes *a* behavior, and *the* behavior $\mathbb{B}$ of the model is the space of all such solutions.

The same behavior may equivalently be described by a model written entirely in terms of the manifest variables, by eliminating all the other variables in the above equations to obtain

$$0 = \frac{w_1}{R} + C\dot{w}_1 - w_2 \tag{6.1}$$

$$0 = -w_3 + L\dot{w}_2 + w_1 \tag{6.2}$$

Still further reduction to a single second-order differential equation is possible, by taking the derivative of one of these equations and eliminating one variable.

Example 6.2 (Mass-Spring System)

An object of mass M moves on a horizontal frictionless slide, and is attached to one end of it by a linear spring with spring constant k. A horizontal force $u(t)$ is applied to the mass. Assume that the variable z measures the change in the spring length from its natural length. From Newton's law we obtain the model

$$M\ddot{z} = -kz + u.$$

Example 6.3 (Inverted Pendulum)

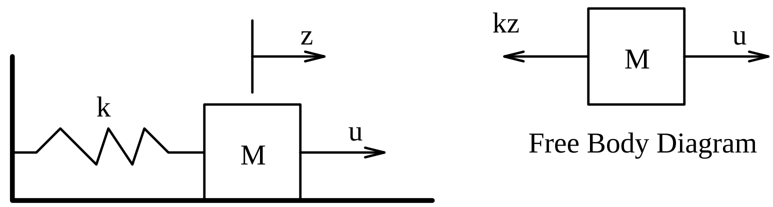


Figure 6.1: Mass Spring System.

A cart of mass M slides on a horizontal frictionless track, and is pulled by a horizontal force $u(t)$. On the cart an inverted pendulum of mass m is attached via a frictionless hinge, as shown in Figure 28.1. The pendulum's center of mass is located at a distance l from its two ends, and the pendulum's moment of inertia about its center of mass is denoted by I. The point of support of the pendulum is a distance $s(t)$ from some reference point. The angle $\theta(t)$ is the angle that the pendulum makes with respect to the vertical axis. The vertical force exerted by the cart on the base of the pendulum is denoted by P, and the horizontal force by N. What we wish to model are the constraints governing the (manifest) signals $u(t)$, $s(t)$ and $\theta(t)$.

First let us write the equations of motion that result from the free-body diagram of the cart. The vertical forces P, R and Mg balance out. For the horizontal forces we have the following equation:

$$M\ddot{s} = u - N. \tag{6.3}$$

From the free-body diagram of the pendulum, the balance of forces in the horizontal direction gives the equation

$$\begin{aligned} m\frac{d^2}{dt^2}\left(s + l\sin(\theta)\right) &= N, \qquad \text{or} \\ m\left(\ddot{s} - l\sin(\theta)(\dot{\theta})^2 + l\cos(\theta)\ddot{\theta}\right) &= N, \end{aligned} \tag{6.4}$$

and the balance of forces in the vertical direction gives the equation

$$\begin{aligned} m\frac{d^2}{dt^2}\left(l\cos(\theta)\right) &= P - mg, \qquad \text{or} \\ m\left(-l\cos(\theta)(\dot{\theta})^2 - l\sin(\theta)\ddot{\theta}\right) &= P - mg. \end{aligned} \tag{6.5}$$

From equations (28.16) and (28.17) we can eliminate the force N to obtain

$$(M+m)\ddot{s} + m\left(l\cos(\theta)\ddot{\theta} - l\sin(\theta)(\dot{\theta})^2\right) = u. \tag{6.6}$$

By balancing the moments around the center of mass, we get the equation

$$I\ddot{\theta} = Pl\sin(\theta) - Nl\cos(\theta). \tag{6.7}$$

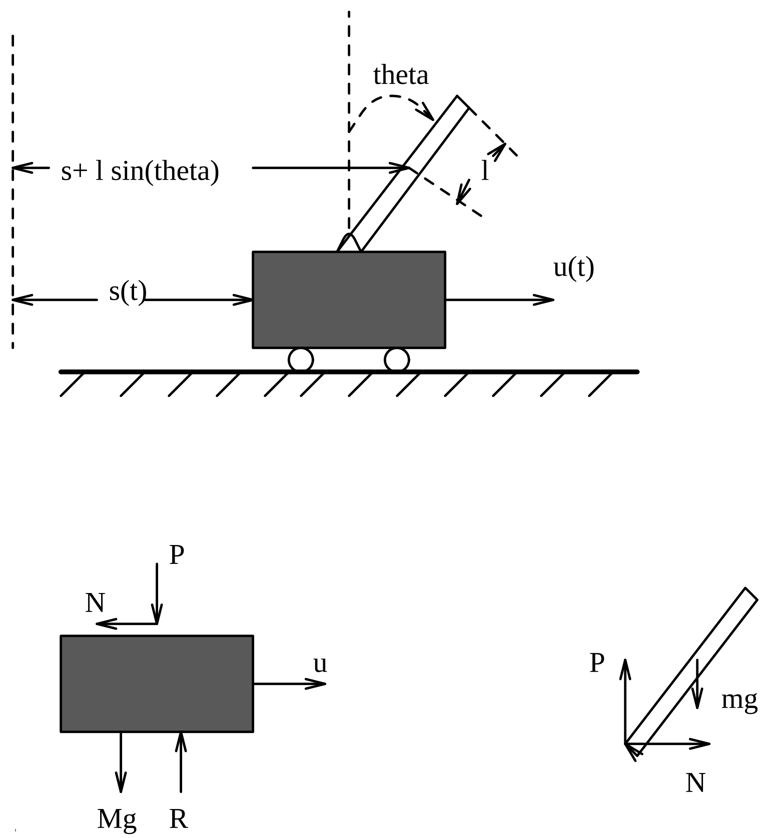


Figure 6.2: Inverted Pendulum

Substituting (28.17) and (28.18) into (28.19) gives us

$$\begin{aligned} I\ddot{\theta} &= l\left(mg - ml\cos(\theta)(\dot{\theta})^2 - ml\sin(\theta)\ddot{\theta}\right)\sin(\theta) \\ &- l\left(m\ddot{s} - ml\sin(\theta)(\dot{\theta})^2 + ml\cos(\theta)\ddot{\theta}\right)\cos(\theta). \end{aligned}$$

Simplifying the above expression gives us the equation

$$(I + ml^2)\ddot{\theta} = mgl\sin(\theta) - ml\ddot{s}\cos(\theta). \tag{6.8}$$

The equations that comprise our model for the system are (28.20) and (28.21).

We can have a further simplification of the system of equations by removing the term $\ddot{\theta}$ from equation (28.20), and the term $\ddot{s}$ from equation (28.21). Define the constants

$$\begin{aligned} \mathcal{M} &= M + m \\ L &= \frac{I + ml^2}{ml}. \end{aligned}$$

Substituting $\ddot{\theta}$ from (28.21) into (28.20), we get

$$\left(1 - \frac{ml}{\mathcal{M}L}\cos(\theta)^2\right)\ddot{s} + \frac{ml}{\mathcal{M}L} g \sin(\theta)\cos(\theta) - \frac{ml}{\mathcal{M}}\sin(\theta)(\dot{\theta})^2 = \frac{1}{\mathcal{M}}u. \quad (6.9)$$

Similarly we can substitute $\ddot{s}$ from (28.20) into (28.21) to get

$$\left(1 - \frac{ml}{\mathcal{M}L}\cos(\theta)^2\right)\ddot{\theta} - \frac{g}{L}\sin(\theta) + \frac{ml}{\mathcal{M}L}\sin(\theta)\cos(\theta)(\dot{\theta})^2 = -\frac{1}{\mathcal{M}L}\cos(\theta)u. \quad (6.10)$$

Example 6.4 (Predator-Prey Model)

While the previous examples are physically based, there are many examples of dynamic models that are hypothesized on the basis of a behavioral pattern. For a classical illustration, consider an island populated primarily by goats and foxes. Goats survive on the island's vegetation while foxes survive by eating goats.

To build a model of the population growth of these two interacting animals, define:

$$\begin{aligned} N_1(t) &= \text{number of goats at time t} && (6.11)\\ N_2(t) &= \text{number of foxes at time t} && (6.12)\end{aligned}$$

where t refers to (discrete) time measured in multiples of months. Volterra proposed the following model:

$$\begin{aligned} N_1(t+1) &= aN_1(t) - bN_1(t)N_2(t) && (6.13)\\ N_2(t+1) &= cN_2(t) + dN_1(t)N_2(t) && (6.14)\end{aligned}$$

The constants a, b, c, and d are all positive, with $a > 1$, $c < 1$. If there were no goats on the island, $N_1(0) = 0$, then — according to this model — the foxes' population would decrease geometrically (i.e. as a discrete-time exponential). If there were no foxes on the island, then the goat population would grow geometrically (presumably there is an unlimited supply of vegetation, water and space). On the other hand, if both species existed on the island, then the frequency of their encounters, which is modeled as being proportional to the product N_1N_2, determines at what rate goats are eaten and foxes are well-fed. Among the questions that might now be asked are: What sorts of qualitative behavioral characteristics are associated with such a model, and what predictions follow from this behavior? What choices of the parameters a, b, c, d best match the behavior observed in practice?

Example 6.5 (Smearing in an Imaging System)

Consider a model that describes the relationship between a two-dimensional object and its image on a planar film in a camera. Due to limited aperture, lens imperfections and focusing errors, the image of a unit point source at the origin in the object, represented by the unit impulse $\delta(x, y)$ in the object plane, will be smeared. The intensity of the light at the image may be modeled by some function $h(x, y)$, $x, y \in \mathbb{R}$, for example $h(x, y) = e^{-a(x^2+y^2)}$. An object $u(x, y)$ can be viewed as the superposition of individual points distributed spatially, i.e.,

$$u(x, y) = \int\int_{-\infty}^{\infty} \delta(x - \lambda, y - \mu)\, u(\lambda, \mu) d\lambda\, d\mu \ .$$

Assuming that the effect of the lens is linear and translation invariant, the image of such an object is given by the following intensity function:

$$m(x, y) = \int\int_{-\infty}^{\infty} h(x - \lambda, y - \mu)\, u(\lambda, d\mu) d\lambda\, d\mu$$

We can view u as the input to this system, m as the output.

6.2 System Representations

There are two general representations of a dynamic model that we shall be interested in, namely behavioral and input-output description.

6.2.1 Behavioral Models

This a very general representation, which we have actually taken as the basis for our initial definition of a dynamic model. In this representation, the system is described as a collection of constraints on designated signals, w_i. Any combination $w(t) = [\ w_1(t), \ \ \cdots, \ \ w_\ell(t)\ ]$ of signals that satisfies the constraints is *a behavior* of the model, $w(t) \in \mathbb{B}$, where $\mathbb{B}$ denotes *the behavior.* An example of such a representation is Example 6.1.

Linearity

We call a model **linear** if its behavior constitutes a vector space, i.e. if *superposition* applies:

$$w_a(t),\ w_b(t) \in \mathbb{B} \implies \alpha\, w_a(t) + \beta\, w_b(t) \in \mathbb{B} \tag{6.15}$$

where α and β are arbitrary scalars. Example 6.1 is evidently linear.

Time-Invariance

We call a model **time-invariant** (or translation-invariant, or shift-invariant) if every possible time shift of a behavior — in which each of the signals is shifted by the same amount — yields a behavior:

$$w(t) \in \mathbb{B} \implies \sigma_\tau w(t) = w(t - \tau) \in \mathbb{B}, \tag{6.16}$$

for all valid τ, i.e. τ for which $\mathbb{T} - \tau \subset \mathbb{T}$, with σ_τ denoting the τ*-shift operator.* Example 6.1 is evidently time-invariant.

Memoryless Models

A model is **memoryless** if the constraints that describe the associated signals $w(\cdot)$ are purely *algebraic*, i.e., they only involve constraints on $w(t_0)$ for each $t_0 \in \mathbb{T}$ (and so do *not* involve derivatives, integrals, etc.). More interesting to us are non-memoryless, or *dynamic* systems, where the constraints involve signal values at different times.

6.2.2 Input-Output Models

For this class of models, the system is modeled as a *mapping* from a set of input signals $u(t)$ to a set of output signals, $y(t)$. We may represent this map as

$$y(t) = (S\ u)\,(t) \tag{6.17}$$

(i.e., the result of operating on the entire signal $u(\cdot)$ with the mapping S yields the signal $y(\cdot)$, and the particular value of the output at some time t is then denoted as above). The above mapping clearly also constitutes a constraint relating $u(t)$ and $y(t)$; this fact could be emphasized by trivially rewriting the equation in the form

$$y(t) - (S\ u)\,(t) = 0\,. \tag{6.18}$$

The definitions of linearity, time-invariance and memorylessness from the behavioral case therefore specialize easily to mappings. An example of a system representation in the form of a mapping is Example 6.5.

Linearity and Time-Invariance

From the behavioral point of view, the signals of interest are given by $w(t) = [u(t)\ y(t)]$. It then follows from the preceding discussion of behavioral models that the model is **linear** if and only if

$$\Big(S\ (\alpha u_a + \beta u_b)\Big)(t)\ = \alpha y_a(t) + \beta y_b(t) = \alpha(S\ u_a)(t) + \beta(S\ u_b)(t) \tag{6.19}$$

and the model is **time-invariant** if and only if

$$\Big(S\ \sigma_\tau u\Big)(t) = (\sigma_\tau y)(t) = y(t-\tau) \tag{6.20}$$

where σ_τ is again the τ-shift operator (so time-invariance of a mapping corresponds to requiring mapping to commute with the shift operator).

Memoryless Models

Again specializing the behavioral definition, we see that a mapping is **memoryless** if and only if $y(t_0)$ only depends on $u(t_0)$, for every $t_0 \in \mathbb{T}$:

$$y(t_0) = (S\ u)\,(t_0)\ = f\Big(u(t_0)\Big)\ . \tag{6.21}$$

Causality

We say the mapping is **causal** if the output does not depend on future values of the input. To describe causality conveniently in mathematical form, define the *truncation operator* P_T on a signal by the condition

$$(P_T\, u)\,(t) = \begin{cases} u(t) & \text{for } t \leq T \\ 0 & \text{for } t > T \end{cases} \quad . \tag{6.22}$$

Thus, if u is a record of a function over all time, then $(P_T\, u)$ is a record of u up to time T, trivially extended by 0. Then the system S is said to be causal if

$$P_T S P_T = P_T S \quad . \tag{6.23}$$

In other words, the output up to time T depends only on the input up to time T.

Example 6.6 Example 6.5 shows a system represented as an input-output map. It is evident that the model is linear, translation-invariant, and not memoryless (unless $h(x, y) = \delta(x, y)$).

Notes

For much more on the behavioral approach to modeling and analysis of dynamic systems, see

J. C. Willems, "Paradigms and Puzzles in the Theory of Dynamic Systems," *IEEE Transactions on Automatic Control,* Vol. 36, pp. 259–294, March 1991.

Exercises

Exercise 6.1 Suppose the output $y(t)$ of a system is related to the input $u(t)$ via the following relation:

$$y(t) = \int_0^{\infty} e^{-(t-s)} u(s) ds.$$

Verify that the model is linear, time-varying, non-causal, and not memoryless.

Exercise 6.2 Suppose the input-output relation of a system is given by

$$y(t) = \begin{cases} u(t) & \text{if } |u(t)| \leq 1 \\ \frac{u(t)}{|u(t)|} & \text{if } |u(t)| > 1 \end{cases} .$$

This input-output relation represents a *saturation* element. Is this map nonlinear? Is it memoryless?

Exercise 6.3 Consider a system modeled as a map from $u(t)$ to $y(t)$, and assume you know that when

$$u(t) = \begin{cases} 1 & \text{for } 1 \leq t \leq 2 \\ 0 & \text{otherwise} \end{cases} ,$$

the corresponding output is

$$y(t) = \begin{cases} e^{t-1} - e^{t-2} & \text{for } t \leq 1 \\ 2 - e^{1-t} - e^{t-2} & \text{for } 1 \leq t \leq 2 \\ e^{2-t} - e^{1-t} & \text{for } t \geq 2 \end{cases} .$$

In addition, the system takes the zero input to the zero output. Is the system causal? Is it memoryless?

A particular mapping that is consistent with the above experiment is described by

$$y(t) = \int_{-\infty}^{\infty} e^{-|t-s|} u(s) ds. \tag{6.24}$$

Is the model linear? Is it time-invariant?

Exercise 6.4 For each of the following maps, determine whether the model is (a) linear, (b) time-invariant, (c) causal, (d) memoryless.

(i)

$$y(t) = \int_0^t (t-s)^3 u(s) ds$$

(ii)

$$y(t) = 1 + \int_0^t (t-s)^3 u(s) ds$$

(iii)

$$y(t) = u^3(t)$$

(iv)

$$y(t) = \int_0^t e^{-ts} u(s) ds$$

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