6.241 Dynamic Systems and Control

Lecture 25: $\mathcal{H}_\infty$ Synthesis

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Standard setup

- Consider the following system, for $t \in R_{\geq 0}$:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_w w(t) + B_u u(t), \quad x(0) = x_0 \\ z(t) &= C_z x(t) + D_{zw} w(t) + D_{zu} u(t) \\ y(t) &= C_y x(t) + D_{yw} w(t) + D_{yu} u(t),\end{aligned}$$

  where

  - w is an exogenous disturbance input (also reference, noise, etc.)
  - u is a control input, computed by the controller K
  - z is the performance output. This is a "virtual" output used only for design.
  - y is the measured output. This is what is available to the controller K

- It is desired to synthesize a controller K (itself a dynamical system), with input y and output u, such that the closed loop is stabilized, and the performance output is minimized, given a class of disturbance inputs.
- In particular, we will look at controller synthesis with $\mathcal{H}_2$ and $\mathcal{H}_\infty$ criteria.

Optimal $\mathcal{H}_\infty$ synthesis?

- In principle, we would like to find a controller K such that minimizes the energy ($\mathcal{L}_2$) gain of the closed-loop system, i.e., that minimizes

$$\|T_{zw}\|_{\mathcal{H}_\infty} = \sup_{w \neq 0} \frac{\|z\|_{\mathcal{L}_2}}{\|w\|_{\mathcal{L}_2}}.$$

- However, the optimal controller(s) are such that $\sigma_{\max}(T_{zw}(j\omega)$ is a constant over all frequencies, the response does not roll off at high frequencies, and the controller is not strictly proper. (The optimal controller is not unique.)

- In addition, computing an optimal controller is numerically challenging.

Sub-optimal $\mathcal{H}_\infty$ synthesis

- A better approach in practice is to pursue a sub-optimal design, i.e., given $\gamma > 0$, find a controller K such that $\|T_{zw}\|_{\mathcal{H}_\infty} < \gamma$, if one exists.
- In other words, assume that the controller K and the disturbance w are playing a zero-sum game, in which the cost is

$$\|z\|^2_{\mathcal{L}_2} - \gamma^2 \|w\|^2_{\mathcal{L}_2}.$$

what is the smallest γ such that the controller can win the game (i.e., achieve a negative cost)?

Approximately optimal $\mathcal{H}_\infty$ synthesis

- The optimal performance γ^* can be approximated arbitrarily well, by a bisection method, maintaining lower and upper bounds $\gamma_- < \gamma^* < \gamma_+$:
  1. Init to, e.g., $\gamma_- = 0$, $\gamma_+ =$ the $\mathcal{H}_\infty$ norm of the $\mathcal{H}_2$ optimal design. Let K_+ be the optimal $\mathcal{H}_2$ controller.
  2. Let $\gamma \leftarrow (\gamma_- + \gamma_+)/2$. Check whether a controller exists such that $\|T_{zw}\|_{\mathcal{H}_\infty} < \gamma$.
  3. If yes, set $\gamma_+ \leftarrow \gamma$, and set K_+ to the controller just designed. Otherwise, set $\gamma_- \leftarrow \gamma$.
  4. Repeat from step 2 until $\gamma_+ - \gamma_- < \epsilon$.
  5. Return K_+.

Simplified setup

- For simplicity, consider the case in which
  - $C_z' D_{zu} = 0$, i.e., the cost if of the form $\int_0^{+\infty} x'Qx + u'Ru \, dt$.
  - $B_w D_{yw}' = 0$, i.e., process noise and sensor noise are uncorrelated.
  - $D_{zu}' D_{zu} = I$, $D_{yw} D_{yw}' = I$.

Full information case (intuition)

- Assume that the state x and the disturbance w are available for measurement, i.e., $y = \begin{bmatrix} x & w \end{bmatrix}'$.
- Assume that the optimal control is of the form $u = F_u x$, and that the optimal disturbance is of the form $w = F_w x$.
- The evolution of the system is completely determined by the initial condition x_0. In particular, defining $A_\infty = A + B_w F_w + B_u F_u$:
  - the energy of the performance output is computed as

$$\|z\|_{\mathcal{L}_2}^2 = \int_0^{+\infty} x_0' \left(e^{A_\infty' t} C_z' C_z e^{A_\infty t} + e^{A_\infty' t} F_u' F_u e^{A_\infty t} \right) x_0 \, dt$$

  - The energy of the disturbance is computed as

$$\|w\|_{\mathcal{L}_2}^2 = \int_0^{+\infty} x_0' e^{A_\infty' t} F_w' F_w e^{A_\infty t} x_0 \, dt.$$

Full information case (intuition)

- Hence the cost of the game is

$$\|z\|_{\mathcal{L}_2}^2 - \gamma^2 \|w\|_{\mathcal{L}_2}^2 = x_0' X_\infty x_0,$$

where X_∞ is the observability Gramian of the pair (C_∞, A_∞), with $C_\infty = \begin{bmatrix} Q^{1/2} & j\gamma F_w' & F_u' \end{bmatrix}'$.

- From the properties of the observability Gramian, it must be the case that

$$A_\infty' X_\infty + X_\infty A_\infty + C_\infty' C_\infty = 0$$

- Assuming that there exist S_u, S_w such that $F_u = S_u X_\infty$ and $F_w = S_w X_\infty$, and expanding, we get

$$\begin{aligned} A' X_\infty + X_\infty S_w' B_w' X_\infty + X_\infty S_u' B_u' X_\infty \\ + X_\infty A + X_\infty B_w S_w X_\infty + X_\infty B_u S_u X_\infty \\ + Q - \gamma^2 X_\infty S_w' S_w X_\infty + X_\infty S_u' S_u X_\infty = 0 \end{aligned}$$

Guess for the structure of the suboptimal controller

- A possible solution would be:

$$A'X_\infty + X_\infty A + C_z'C_z = X_\infty(B_uB_u' - \gamma^{-2}B_wB_w')X_\infty,$$

$$F_u = -B_u'X_\infty, \qquad F_w = \frac{1}{\gamma^2}B_w'X_\infty$$

- This is a Riccati equation, but notice that the quadratic term is not necessarily sign definite.
- Similar considerations hold for the "observer" Riccati equation

$$AY_\infty + Y_\infty A' + B_w'B_w = Y_\infty(C_yC_y' - \gamma^{-2}C_zC_z')Y_\infty$$

- The observer gain would be

$$L = -(I - \gamma^{-2}Y_\infty X_\infty)^{-1}Y_\infty C_y'.$$

Note the inversion of the matrix $\gamma^2 I - Y_\infty X_\infty$.

Suboptimal $\mathcal{H}_\infty$ controller

- Assuming the following technical conditions hold:
  - (A, B_u) stabilizable, (C_y, A) detectable.
  - The matrices $\begin{bmatrix} A - j\omega I & B_w \end{bmatrix}$, $\begin{bmatrix} A' - j\omega I & C_z' \end{bmatrix}$ must have full row rank.
- A controller K such that $\|z\|^2_{\mathcal{L}_2} - \gamma^2 \|w\|^2_{\mathcal{L}_2} < 0$ exists if and only if
  1. The following Riccati equation has a stabilizing solution $X_\infty \geq 0$:
     $$A'X_\infty + X_\infty A + C_z'C_z = X_\infty(B_u B_u' - \gamma^{-2} B_w B_w')X_\infty,$$
  2. The following Riccati equation has a stabilizing solution $Y_\infty \geq 0$:
     $$AY_\infty + Y_\infty A' + B_w' B_w = Y_\infty(C_y C_y' - \gamma^{-2} C_z C_z')Y_\infty$$
  3. The matrix $\gamma^2 I - Y_\infty X_\infty$ is positive definite.

Current research

- Distributed control systems
- Networked control systems (quantization, bandwidth limitations, etc.)
- Computational methods
- Nonlinear systems/robustness (ISS, IQCs, polynomial systems, SoS, etc.)
- Hybrid/switched systems
- System ID/Model reduction
- Robust/Adaptive control

Other classes

- 6.231 Dynamic Programming and Stochastic Control
- 6.242 Advanced Linear Control Systems
- 6.243 Dynamics of Nonlinear Systems
- 6.245 Multivariable Control Systems
- 6.256 Algebraic Techniques and Semidefinite Optimization
- 6.246-7 Advanced Topics in Control
- 2.152 Nonlinear Control System Design
- 10.552 Advanced Systems Engineering (R. Braatz on LMIs for optimal/robust control)
- 16.322 Stochastic Estimation and Control
- 16.323 Principles of Optimal Control
- 16.333 Aircraft Stability and Control

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