

# 6.241 Dynamic Systems and Control

## Lecture 24: $\mathcal{H}_2$ Synthesis

Emilio Frazzoli

Aeronautics and Astronautics  
Massachusetts Institute of Technology

May 4, 2011

# Standard setup

- Consider the following system, for  $t \in \mathbb{R}_{\geq 0}$ :

$$\dot{x}(t) = Ax(t) + B_w w(t) + B_u u(t), \quad x(0) = x_0$$

$$z(t) = C_z x(t) + D_{zw} w(t) + D_{zu} u(t)$$

$$y(t) = C_y x(t) + D_{yw} w(t) + D_{yu} u(t),$$

where

- $w$  is an exogenous **disturbance input** (also reference, noise, etc.)
- $u$  is a **control** input, computed by the controller  $K$
- $z$  is the **performance output**. This is a “virtual” output used only for design.
- $y$  is the **measured output**. This is what is available to the controller  $K$
- It is desired to **synthesize a controller**  $K$  (itself a dynamical system), with input  $y$  and output  $u$ , such that the closed loop is stabilized, and the performance output is minimized, given a class of disturbance inputs.
- In particular, we will look at controller synthesis with  $\mathcal{H}_2$  and  $\mathcal{H}_\infty$  criteria.

# Interpretation of the $\mathcal{H}_2$ norm — deterministic

- Consider a stable, causal CT LTI system with state-space model  $(A, B, C, D)$ , transfer function  $G(s)$ , and impulse response  $G(t)$ .
- The  $\mathcal{H}_2$  norm of  $G$  measures:

A) The energy of the impulse response:

$$\begin{aligned}\|G\|_{\mathcal{L}_2}^2 &:= \sum_i \sum_j \int_0^{+\infty} \|g_{ij}(t)\|_2^2 dt = \int_0^{+\infty} \|G(t)\|_F^2 dt \\ &= \text{Tr} \left[ \int_0^{+\infty} G(t)' G(t) dt \right] = \frac{1}{2\pi} \text{Tr} \left[ \int_{-\infty}^{+\infty} G(j\omega)' G(j\omega) d\omega \right] =: \|G\|_{\mathcal{H}_2}^2.\end{aligned}$$

B) The energy of the response to initial conditions, of the form  $x(0) = Bu_0$ , for  $u_0 = (1, 1, \dots, 1)'$ . Set  $u(t) = u_0\delta(t)$  to see this.

- Clearly, in order for  $\|G\|_{\mathcal{L}_2} = \|G\|_{\mathcal{H}_2}$  to be finite, it is necessary that  $\lim_{\omega \rightarrow \infty} G(j\omega) = 0$ , i.e., that the system is strictly causal  $\Leftrightarrow D = 0$ .

# Interpretation of the $\mathcal{H}_2$ norm — stochastic

- Consider a stable, strictly causal CT LTI system with state-space model  $(A, B, C, 0)$ , transfer function  $G(s)$ , and impulse response  $G(t)$ .
- Consider a hypothetical stochastic input signal  $u$  such that  $\mathbb{E}[u(t)] = 0$ , and  $\mathbb{E}[u(t)u(t + \tau)'] = I\delta(\tau)$ . This is called white noise, and is just a mathematical abstraction, since it is a signal with infinite power.
- The  $\mathcal{H}_2$  norm of  $G$  measures:
  - C) The (expected) power of the response to white noise:

$$\begin{aligned} & \mathbb{E} \left[ \lim_{T \rightarrow +\infty} \frac{1}{T} \text{Tr} \left( \int_0^T y(t)y(t)' dt \right) \right] \\ &= \lim_{T \rightarrow +\infty} \frac{1}{T} \text{Tr} \left( \int_0^T \mathbb{E} \left[ \int_0^t \int_0^t G(t - \tau_1)u(\tau_1)u(\tau_2)'G(t - \tau_2)' d\tau_1 d\tau_2 \right] dt \right) \\ &= \lim_{T \rightarrow +\infty} \frac{1}{T} \text{Tr} \left( \int_0^T \int_0^t G(t - \tau)G(t - \tau)' d\tau dt \right) \\ &= \lim_{T \rightarrow +\infty} \text{Tr} \left[ \int_0^T G(T - \tau)G(T - \tau)' d(T - \tau) \right] = \|G\|_{\mathcal{L}_2}^2 = \|G\|_{\mathcal{H}_2}^2. \end{aligned}$$

# Computation of the $\mathcal{H}_2$ norm

- Computation of the  $\mathcal{H}_2$  norm is easy through state-space methods. In fact,

$$\|G\|_{\mathcal{H}_2}^2 = \text{Tr} \left[ \int_0^{+\infty} G(t)' G(t) dt \right] = \text{Tr} \left[ \int_0^{+\infty} G(t) G(t)' dt \right].$$

- Since  $G(t) = Ce^{At}B$  (recall that  $D = 0$  is necessary for the  $\mathcal{H}_2$  norm to be finite), we get

$$\begin{aligned} \|G\|_{\mathcal{H}_2}^2 &= \text{Tr} \left[ \int_0^{+\infty} B' e^{A't} C' C e^{At} B dt \right] = \text{Tr} [B'QB] \\ &= \text{Tr} \left[ \int_0^{+\infty} C e^{At} B B' e^{A't} C' dt \right] = \text{Tr} [CPC'], \end{aligned}$$

where

- $Q$  is the observability Gramian, satisfying  $AQ + AQ' = -CC'$ .
- $P$  is the reachability Gramian, satisfying  $A'P + AP = -B'B$ .

# Structure of the $D$ block

We will make the following assumptions on the structure of the  $D$  block:

- $D_{yu} = 0$ .
  - We can always make this assumption, since  $u$  is known.
- $D_{zw} = 0$ 
  - The  $\mathcal{H}_2$  norm of a system that is not strictly proper (i.e., such that  $\lim_{s \rightarrow \infty} G(s) = G_\infty > 0$ ) is  $+\infty$ .
  - Note that  $T_{zw}(\infty) = D_{zw} + D_{zu}D_{uy}D_{yw}$ . If there is no  $D_{uy}$  such that  $T_{zw}(\infty) = 0$ , then the problem is ill-posed. If there is one such  $D_{uy}^0$ , then define  $\tilde{u} = u + D_{uy}^0 y$ , and rewrite the problem as follows:

$$\begin{aligned}\tilde{A} &= A + B_u D_{yu}^0 C_y \\ \tilde{B}_w &= B_w + B_u D_{uy}^0 D_{uw} \\ \tilde{C}_z &= C_z + D_{zu} D_{uy}^0 C_y \\ \tilde{D}_{zw} &= D_{zw} + D_{zu} D_{uy}^0 D_{yw} = 0.\end{aligned}$$

# The LQR problem

- The LQR problem is the special case of  $\mathcal{H}_2$  synthesis in which we assume:
  - Full state feedback:  $C_y = I$ ;
  - No disturbance input:  $w = 0$ .
- Objective: find a control signal  $u(t, x) \in \mathcal{L}_2$  that minimizes

$$\|z\|_2^2 = \int_0^{+\infty} \|C_z x + D_{zu} u\|_2^2 dt,$$

given the initial condition  $x(0)$ .

- Note that if  $C_z = [\sqrt{Q} \ 0]'$  and  $D_{zu} = [0 \ \sqrt{R}]'$  then we get

$$\|z\|_2^2 = \int_0^{+\infty} (x' Q x + u' R u) dt,$$

which is the “usual” way the LQR problem is formulated.

# Towards a solution of the LQR problem (intuition)

- Consider a **stabilizing** control law of the form  $u = FX$ , and assume  $C'_z D_{zu} = 0$ . By assumption,  $A_F = A + B_u F$  is stable.

$$\|z\|_2^2 = \int_0^{+\infty} x_0' \left( e^{A_F t} C'_z C_z e^{A_F t} + e^{A_F t} F' D'_{zu} D_{zu} F e^{A_F t} \right) x_0 dt,$$

i.e.,  $\|z\|_2^2 = x_0' X_F x_0$ , where  $X_F$  is the observability gramian of the pair  $(C_F, A_F)$ , with  $C_F = C_z + D_{zu} F$ , and

$$A'_F X_F + X_F A_F = -C'_F C_F.$$

- Since we know that the closed-loop is stable, we can also rewrite the above equation as

$$\|z\|_2^2 = \int_0^{+\infty} \frac{d}{dt} (x(t)' X_F x(t)) dt.$$

- The integrand can be written as

$$x(t)' (A' X_F + F' B'_u X_F + X_F A + X_F B_u F) x(t)$$

# Towards a solution of the LQR problem (intuition)

- Assume there is a matrix  $S$  such that  $F = SX_F$ . Then, the integrand becomes

$$\begin{aligned}x(t)' (A'X_F + X_F S' B'_u X_F + X_F A + X_F B_u S X_F) x(t) \\ = -x(t)' (C'_z C_z + X_F S' D'_{zu} D_{zu} S X_F) x(t)\end{aligned}$$

- In other words, it must be that

$$A'X_F + X_F A + X_F S' B'_u X_F + X_F B_u S X_F + C'_z C_z + X_F S' D'_{zu} D_{zu} S X_F = 0$$

- Set  $S = -(D'_{zu} D_{zu})^{-1} B'_u$ . Then,  $X_F$  must satisfy

$$A'X_F + X_F A + X_F B_u (D'_{zu} D_{zu})^{-1} B'_u X_F + C'_z C_z = 0$$

and

$$F = -(D'_{zu} D_{zu})^{-1} B'_u X_F$$

- Is this “solution” indeed stabilizing/optimal?

# On Riccati equations

- We have already encountered a matrix equation that plays a major role in control, i.e., the (continuous-time) Lyapunov equation:

$$A'X + XA + Q = 0.$$

- This equation can be used, among other things, to check stability of a LTI system, and to compute reachability/observability gramians.
- The Lyapunov equation is linear in  $X$ , and can be easily solved.
- Another important equation in control theory is the (c.t.) algebraic Riccati equation:

$$A'X + XA + XRX + Q = 0.$$

- The Riccati equation is quadratic in  $X$ ; what can we say about its solutions, and how do we compute them?

# Hamiltonian matrices

- It turns out that to each Riccati equation we can associate a **Hamiltonian matrix** of the form

$$H := \begin{bmatrix} A & R \\ -Q & -A' \end{bmatrix},$$

which will be used to compute solutions to the Riccati equation.

- The spectrum of  $H$  is symmetric with respect to the imaginary axis. To see this, consider the similarity transformation:

$$\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}^{-1} \begin{bmatrix} A & R \\ -Q & -A' \end{bmatrix} \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} R & -A \\ -A' & Q \end{bmatrix} = \begin{bmatrix} -A' & Q \\ -R & A \end{bmatrix} = -H'.$$

- In other words,  $H$  and  $-H'$  are similar, and hence if  $\lambda$  is an eigenvalue of  $H$  so is  $-\lambda'$ .

# Computing solutions to the Riccati equation

- Assume that  $H$  has no eigenvalues on the imaginary axis. Then  $H$  will have  $n$  eigenvalues in the open left half plane, and  $n$  in the open right half plane.
- Let  $\mathcal{X}_-$  be the subspace spanned by the eigenvectors associated with the eigenvalues with negative real part, and find  $n \times n$  matrices  $X_1$  and  $X_2$  such that  $\mathcal{X}_- = \text{Ra} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ .
- If  $X_1$  is nonsingular, then set  $X := X_2 X_1^{-1}$ .
- Note that  $X$  is unique, since any other set of basis vectors satisfies  $\tilde{X}_1 = X_1 S$ ,  $\tilde{X}_2 = X_2 S$ , for some invertible matrix  $S$ , and  $X := \tilde{X}_2 \tilde{X}_1^{-1} = X_2 S S^{-1} X_1$ .

## Theorem

Assume that (i)  $H$  has no eigenvalues on the imaginary axis, and that (ii) the matrix  $X_1$  in the above construction is not singular. Then,

- 1  $X$  is real symmetric;
- 2  $X$  satisfies the Riccati equation  $A'X + XA + XRX + Q = 0$
- 3 All the eigenvalues of the matrix  $A + RX$  are in the open left half plane.

# Computing solutions to the Riccati equation—proof

- $X = X_2 X_1^{-1}$  is real symmetric.

- Note that there exists a stable  $n \times n$  matrix  $H_-$  such that

$$H \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} H_-$$

- Premultiply by  $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}' \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$ :

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}' \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} H \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}' \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} H_-$$

- The left hand side is Hermitian—so the right hand side is also Hermitian, and

$$(-X_1' X_2 + X_2' X_1) H_- + H_-' (-X_1' X_2 + X_2' X_1) = 0.$$

- This is a Lyapunov equation, and since  $H_-$  is stable, it has a unique solution  $-X_1' X_2 = X_2' X_1$ .
- Hence the matrix  $X := X_2 X_1^{-1} = (X_1^{-1})' (X_1' X_2) X_1^{-1}$  is Hermitian. Since  $X_1$  and  $X_2$  can be chosen to be real, and  $X$  is unique,  $X$  is real and symmetric.

# Computing solutions to the Riccati equation—proof

- $X$  satisfies the Riccati equation  $A'X + XA + XRX + Q = 0$ :

- Start with  $H \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} H_-$ , and left-/right-multiply as follows:

$$\begin{bmatrix} X & -I \end{bmatrix} \left( H \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \right) X_1^{-1} = \begin{bmatrix} X & -I \end{bmatrix} \left( \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} H_- \right) X_1^{-1},$$

$$\begin{bmatrix} X & -I \end{bmatrix} \begin{bmatrix} A & R \\ -Q & -A' \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} X_2 & -X_2 \end{bmatrix} H_- X_1^{-1}$$
$$XA + Q + XRX + A'X = 0.$$

- $A + RX$  is stable:

- Similarly,

$$\begin{bmatrix} I & 0 \end{bmatrix} \left( H \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \right) X_1^{-1} = \begin{bmatrix} I & 0 \end{bmatrix} \left( \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} H_- \right) X_1^{-1},$$
$$A + RX = X_1 H_- X_1^{-1},$$

i.e.,  $A + RX$  is similar to a stable matrix, and hence stable.

# Technical conditions

- A1)  $(A, B_u)$  stabilizable.
- A2)  $(C_z, A)$  detectable.
- A3)  $\begin{bmatrix} A - j\omega I & B_u \\ C_z & D_{zu} \end{bmatrix}$  has full column rank for all  $\omega \in \mathbb{R}$
- A4)  $D'_{zu} D_{zu} = R$ , invertible, i.e.,  $D_{zu}$  has full column rank.

A1-A3) ensure that the Riccati equation admits a solution  $X$  that is positive semi-definite. In particular, A1) is obviously necessary, A2) ensures that any unstable mode of  $A$  will be detected by the performance output, and A3) ensures that the control effort is penalized at all frequencies (this is an additional technical condition ensuring that the Hamiltonian does not have purely imaginary eigenvalues). A4) is just for convenience.

# LQR: optimal control

- It turns out that the optimal controller can be obtained from the unique, symmetric, positive-definite solution  $X$  of the (algebraic) Riccati Equation

$$(A - B_u R_{uu}^{-1} D'_{zu} C_z)' X + X(A - B_u R_{uu}^{-1} D'_{zu} C_z) - X B_u R_{uu}^{-1} B_u' X + C_z'(I - D_{zu} R_{uu}^{-1} D'_{zu}) C_z = 0$$

by setting

$$F = -R_{uu}^{-1}(B_u' X + D'_{zu} C_z).$$

- Define  $A_F := A + B_u F$ , and  $C_F := C_z + D_{zu} F$ . Recall that  $X$  can be interpreted as the observability Gramian of  $(A_F, C_F)$ , describing the energy of the impulse response of the closed-loop system  $(A_F, I, C_F, 0)$ .
- Hence  $\|z\|_2^2 = x_0' X x_0$ .

## Note on the detectability of $(C_z, A)$

- **Claim:** Since  $(C_z, A)$  is detectable, if  $u, z \in \mathcal{L}_2$ , then  $x \in \mathcal{L}_2$ , and  $x \rightarrow 0$ .
- **Proof:** Design a hypothetical observer using  $z$  to compute an estimate  $\hat{x}$  of the state  $x$ . Then

$$\dot{\hat{x}} = A\hat{x} + B_u u + L(C_z \hat{x} - z + D_{zu} u) = (A + LC_z)\hat{x} + (B_u + LD_{zu})u - Lz,$$

and hence  $\hat{x} \in \mathcal{L}_2$ ,  $\hat{x} \rightarrow 0$ . Moreover, since the observer is stable,  $x - \hat{x} \rightarrow 0$ .

# Optimality of the proposed control law

- Assume that  $u = Fx + v$ . Then one can write

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} = \begin{bmatrix} A_F & B_u \\ C_F & D_{zu} \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}, \quad x(0) = x_0$$

- Note that  $v \in \mathcal{L}_2 \Rightarrow x, z, u \in \mathcal{L}_2$  (stability of  $A_F$ ), and  $u, z \in \mathcal{L}_2 \Rightarrow v, x \in \mathcal{L}_2$  (detectability of the state in the performance output). So minimizing over  $u \in \mathcal{L}_2$  is equivalent to minimizing over  $v \in \mathcal{L}_2$ .
- Differentiate  $x(t)'Xx(t)$  along system trajectories, noting that  $C_F' D_{zu} = -XB_u$ :

$$\begin{aligned} \frac{d}{dt} x' X x &= x'(A_F' X + X A_F)x + 2x' X B_u v \\ &= -x' C_F' C_F x - 2x' C_F' D_{zu} v - v' D_{zu}' D_{zu} v + v' D_{zu}' D_{zu} v \\ &= -|z|^2 + v' R v. \end{aligned}$$

- Integrating from 0 to  $+\infty$ , we get

$$-x_0' X x_0 = -\|z\|_2^2 + \|\sqrt{R}v\|_2^2.$$

Hence the minimum is attained for  $v = 0$ .

# LQE problem — Kalman filter

- The LQE problem is the special case of  $\mathcal{H}_2$  synthesis addressing the design of an observer (i.e.,  $u$  takes the role of the observer update), assuming
  - Full state updates:  $B_u = I$ .
  - Zero initial conditions:  $\tilde{x}(0) = x(0) - \hat{x}(0) = 0$ .
- Objective: find an update signal  $u(t, \tilde{x}) \in \mathcal{L}_2$  that minimizes the power in the error signal due to white noise disturbance  $w$ .
- Note that the disturbance enters the system in two places:
  - As process noise:  $\dot{\tilde{x}} = A\tilde{x} + B_w w$ .
  - As sensor noise:  $y = C_y \tilde{x} + D_{yw} w$ .
- If  $C_y = [\sqrt{Q} \quad 0]'$  and  $D_{yw} = [0 \quad \sqrt{R}]$ , then the process noise and sensor noise are not correlated,

$$\mathbb{E}[w' B_w' D_{yw} w'] = 0,$$

which is the “usual” way the LQE problem is formulated.

# Towards a solution of the LQE problem (intuition)

- Consider a **stabilizing** update law of the form  $u = L(C_y \tilde{x} + D_{yw} w)$ , and assume  $B_w D'_{yw} = 0$ . By assumption,  $A_L = A + LC_y$  is stabilizing.
- The power of the error, under white noise disturbance, is

$$P_z = \text{Tr} \left[ \int_0^{+\infty} (B_w + LD_{yw})' e^{A'_L t} e^{A_L t} (B_w + LD_{yw}) dt \right],$$

i.e.,  $P_z = \text{Tr}[Y_L]$ , where  $Y_L$  is the controllability gramian of the pair  $(A_L, B_L)$ , with  $B_L = B_w + LD_{yw}$ , and

$$A_L Y_L + Y_L A'_L = -B_L B'_L.$$

- In other words,

$$A Y_L + LC_y Y_L + Y_L A' + Y_L C'_y L' + B_w B'_w + LD_{yw} D'_{yw} L' = 0$$

# Towards a solution of the LQE problem (intuition)

- Assume there is a matrix  $S$  such that  $L = Y_L S$ ; then,  $Y_L$  must satisfy

$$AY_L + Y_L S C_y Y_L + Y_L A' + Y_L C_y' S' Y_L' + B_w B_w' + Y_L S D_{yw} D_{yw}' S' Y_L' = 0$$

- Set  $S = -C_y'(D_{yw}D_{yw}')^{-1}$ . Then,  $Y_L$  must satisfy

$$AY_L + Y_L A' - Y_L C_y'(D_{yw}D_{yw}')^{-1}C_y Y_L + B_w B_w' = 0,$$

and

$$L = -Y_L C_y'(D_{yw}D_{yw}')^{-1}.$$

- Duality to the LQR problem is more and more apparent...

# Technical conditions

- B1)  $(C_y, A)$  detectable.
- B2)  $(A, B_w)$  stabilizable.
- B3)  $\begin{bmatrix} A - j\omega I & B_w \\ C_y & D_{yw} \end{bmatrix}$  has full row rank for all  $\omega \in \mathbb{R}$ .
- B4) Assume  $D_{yw}D'_{yw} = R_{ww}$  invertible, i.e.,  $D_{yw}$  has full row rank.

B1-B3) ensure that the Riccati equation admits a solution  $Y$  that is positive semi-definite. In particular, B1) is obviously necessary, B2) ensures that any unstable mode of  $A$  can be excited by the disturbance, and B3) ensures that errors are penalized at all frequencies (this is an additional technical condition ensuring that the Hamiltonian does not have purely imaginary eigenvalues). B4) is just for convenience.

# LQE: optimal observer

- It turns out that the optimal observer can be obtained from the unique, symmetric, positive-definite solution  $Y$  of the (algebraic) Riccati Equation

$$(A - B_w D'_{yw} R_{ww}^{-1} C_y)' Y + Y(A - B_w D'_{yw} R_{ww}^{-1} C_y) - Y C_y R_{ww}^{-1} C_y' Y + B_w (I - D'_{yw} R_{ww}^{-1} D_{yw}) B_w' = 0$$

by setting

$$L = -(Y C_y + B_w D'_{yw}) R_{ww}^{-1}.$$

- Define  $A_L := A + L C_y$ , and  $B_L := B_w + L D_{yw}$ . Recall that  $Y$  can be interpreted as the reachability Gramian of  $(A_L, B_L)$ , describing the power of the response to a white noise input of the closed-loop system  $(A_L, B_L, I, 0)$ .
- Hence  $P_z = Y$ .
- Optimality is proven in a similar way as that of LQR.

## $\mathcal{H}_2$ Synthesis — LQG

- The general version of the problem can be seen as a combination of the LQR problem and of the LQE problem. This is also called the LQG problem.
- By the separation principle, we can design the optimal controller for LQR, and independently design the optimal observer for LQE.
- Can we claim that the model-based output feedback controller is indeed optimal?

# Technical conditions

- A1, B1)  $(A, B_u)$  stabilizable,  $(C_y, A)$  detectable.
- A3)  $\begin{bmatrix} A - j\omega I & B_u \\ C_z & D_{zu} \end{bmatrix}$  has full column rank for all  $\omega \in \mathbb{R}$ .
- B3)  $\begin{bmatrix} A - j\omega I & B_w \\ C_y & D_{yw} \end{bmatrix}$  has full row rank for all  $\omega \in \mathbb{R}$ .
- A4, B4)  $D'_{zu}D_{zu} = R_{uu} > 0$ ,  $D_{yw}D'_{yw} = R_{ww} > 0$ .

## $\mathcal{H}_2$ optimal controller

- Controller gain:  $F = -R_{uu}^{-1}(B_u'X + D_{zu}'C_z)$ , where  $X$  is the stabilizing solution to the ARE:

$$(A - B_u R_{uu}^{-1} D_{zu}' C_z)' X + X(A - B_u R_{uu}^{-1} D_{zu}' C_z) - X B_u R_{uu}^{-1} B_u' X + C_z'(I - D_{zu} R_{uu}^{-1} D_{zu}') C_z = 0.$$

- Observer gain:  $L = -(Y C_y' + B_w D_{yw}') R_{ww}^{-1}$ , where  $Y$  is the stabilizing solution to the ARE:

$$(A - B_w D_{yw}' R_{ww}^{-1} C_y) Y + Y(A - B_w D_{yw}' R_{ww}^{-1} C_y)' - Y C_y' R_{ww}^{-1} C_y Y + B_w(I - D_{yw}' R_{ww}^{-1} D_{yw}) B_w' = 0.$$

- Controller/Observer models:

$$G_c := \begin{bmatrix} A + B_u F & I \\ C_z + D_{zu} F & 0 \end{bmatrix}, \quad G_f := \begin{bmatrix} A + L C_y & B_w + L D_{yw} \\ I & 0 \end{bmatrix}.$$

## $\mathcal{H}_2$ optimal controller

- The state-space model of the optimal controller is then given by

$$K = \left[ \begin{array}{c|cc} A + B_u F + L C_y & -L & B_u \\ \hline F & 0 & I \\ -C_y & I & 0 \end{array} \right],$$

where the second input and second output of  $K$  are connected through an arbitrary stable system  $Q$ .

- Lengthy calculations show that

$$\|C_{zw}\|_{\mathcal{H}_2}^2 = \text{Tr}[B'_w X B_w] + \text{Tr}[D_{zu} F Y F' D'_{zu}] + \text{Tr}[R_{uu} Q R_{ww}].$$

- Clearly, the minimum amplification is attained when  $Q = 0$ , i.e., the conjectured model-based output feedback controller is indeed optimal.

MIT OpenCourseWare  
<http://ocw.mit.edu>

## 6.241J / 16.338J Dynamic Systems and Control

Spring 2011

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.