6.241 Dynamic Systems and Control

Lecture 23: Feedback Stabilization

Readings: DDV, Chapter 28

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Stabilization

- The state of a reachable system can be steered to any desired state in finite time, even if the system is "unstable."
- However, an "open-loop" control strategy depends critically on a number of assumptions:
  - Perfect knowledge of the model;
  - Perfect knowledge of the initial condition;
  - No input constraints.
- It is necessary to use some information on the actual system state in the computation of the control input: i.e., feedback.
- Feedback can also improve the performance of stable systems... but done incorrectly, can also make things worse, most notably, make stable systems unstable.

State Feedback

- Assume we can measure all components of a system's state, i.e., consider a state-space model of the form $(A, B, I, 0)$.
- Assume a linear control law of the form $u = Fx + v$.
- In CT, the closed-loop system model is $(A + BF, B, I, 0)$.
- Hence, it is clear that the closed-loop system is stable if and only if the eigenvalues of $A - BF$ are all in the open left-half plane (or all inside the unit circle, in the DT time case).

Eigenvalue Placement

Theorem

There exists a matrix F such that

$$\det(\lambda I - (A + BF)) = \prod_{i=1}^{n} (\lambda - \mu_i)$$

for any arbitrary self-conjugate set of complex numbers $\mu_1, \ldots, \mu_n \in \mathbb{C}$ if and only if (A, B) is reachable.

Proof (necessity):

- Suppose λ_i is an unreachable mode, and let w_i be the associated left eigenvector. Hence, $w_i^T A = \lambda_i w_i^T$, and $w_i^T B = 0$.
- Then,

  $$w_i^T (A + BF) = w_i^T A + w_i^T BF = \lambda_i w_i^T + 0,$$

  i.e., λ_i is an eigenvalue of $A + BF$ for any F!

Eigenvalue Placement

Proof — Sufficiency:

- Assuming the system is reachable, find a feedback such that the closed-loop poles are at the desired locations. We will prove this only for the single-input case.

- If the system is reachable, then w.l.g. we can assume its realization is in the controller canonical form: the coefficients of the characteristic polynomial are a_1, a_2, ... a_n.

- The coefficients of the closed-loop characteristic polynomial are $(a_1 - f_1)$, etc.

- Just choose $f_i = a_i - a_i^d$, $i = 1, \ldots, n$.

Ackermann Formula

- $$F = -[0, 0, \ldots, 1] R_n^{-1} \alpha^d(A).$$

Observers

- What if we cannot measure the state?
- Design a model-based observer, i.e., a system that contains a simulation of the system, and tries to match its state.

$$d\hat{x}/dt = A\hat{x} + Bu - L(y - \hat{y}).$$

- Error dynamics: $e = x - \hat{x}$:

$$\dot{e} = \dot{x} - \dot{\hat{x}} = Ax + Bu - A\hat{x} - Bu + L(y - \hat{y}) = (A + LC)e.$$

- Same results (dual) as for reachability.

Eigenvalue placement

Theorem

There exists a matrix L such that

$$\det(\lambda I - (A + LC)) = \prod_{i=1}^{n} (\lambda - \mu_i)$$

for any arbitrary self-conjugate set of complex numbers $\mu_1, \ldots, \mu_n \in \mathbb{C}$ if and only if (C, A) is observable.

- Ackermann formula:

$$L = -\alpha^d(A) O_n^{-1} \begin{bmatrix} 0 \\ 0 \\ \ldots \\ 1 \end{bmatrix}$$

Model-based output-feedback controller

—model-based controller block diagram—

- We have

$$\dot{x} = Ax + Bu = Ax + B(r + F\hat{x}) = Ax + BF\hat{x} + Br$$

- now define $\tilde{x} = x - \hat{x}$:

$$\dot{x} = (A + BF)x - BF\tilde{x} + Br$$

- In summary:

$$\frac{d}{dt}\begin{bmatrix} x \\ \tilde{x} \end{bmatrix} = \begin{bmatrix} A + BF & -BF \\ 0 & A + LC \end{bmatrix}\begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r.$$

Synthesis of model-based output feedback controller

- Poles of the closed-loop $=$ c.l. poles of the controller $\cup$ c.l. poles of the observer.

- Separation principle: can design controller and observer independently!

Parameterization of all stabilizing controllers (SISO case)

- Consider the feedback interconnection of a plant G and a controller K.
- Write the plant transfer function as $G(s) = N(s)/M(s)$, and the controller transfer function as $K(s) = Y(s)/X(s)$.
- This can always be done in such a way that $N(s)$, $M(s)$, and $Y(s)$, $X(s)$ are stable transfer functions—even in the case in which G and/or K are themselves unstable.
- The closed-loop system is (externally) stable if and only if

$$\frac{G(s)}{1+K(s)G(s)}, \qquad \frac{K(s)}{1+K(s)G(s)}, \qquad \frac{K(s)G(s)}{1+K(s)G(s)}$$

  are stable transfer functions.
- Note that the transfer functions above can be rewritten as:

$$\frac{N(s)X(s)}{D(s)}, \qquad \frac{M(s)Y(s)}{D(s)}, \qquad \frac{N(s)Y(s)}{D(s)}.$$

  where $D(s) = M(s)X(s) - N(s)Y(s)$.

Bezout's identity

- In other words, since the terms appearing at the numerators are all products of stable transfer functions (and hence stable transfer functions), a necessary and sufficient condition for the stability of the interconnection is that $1/D(s)$ is a stable transfer function.

- In other words, $D(s) = M(s)X(s) - N(s)Y(s)$ must have no zeroes in the open left half-plane (CT), or in the unit disk (DT).

- It turns out that one can, without loss of generality [1], set $D(s) = 1$, in which case we get the so-called Bezout's identity

$$M(s)X(s) - N(s)Y(s) = 1.$$

[1] You can see this by writing $Y'(s) = Y(s)/D(s)$, and $X'(s) = X(s)/D(s)$. Clearly, this is still a valid way of expressing $K(s)$, i.e., $K(s) = Y'(s)/X'(s)$, and both $Y'(s)$ and $X'(s)$ are stable transfer functions. Writing down the stability condition in this case and simplifying, you get Bezout's identity.

Youla's Q parameterization

Theorem

Let $G(s) = N(s)/M(s)$, and let $K_0(s) = Y_0(s)/X_0(s)$, with $N(s)$, $M(s)$, $Y_0(s)$, and $X_0(s)$ stable transfer functions, be a stabilizing feedback controller, and such that

$$M(s)X_0(s) - N(s)Y_0(s) = 1.$$

Then all feedback stabilizing controllers for G are given by

$$K(s) = \frac{Y_0(s) - M(s)Q(s)}{X_0(s) - N(s)Q(s)},$$

where $Q(s)$ is an arbitrary stable transfer function.

Note that with this parameterization, the I/O transfer functions are affine in Q:

- $r \to y$: $N(s)(X_0(s) - N(s)Q(s))$;
- $d \to u$: $M(s)(Y_0(s) - M(s)Q(s))$;
- $d \to y$: $M(s)(X_0(s) - N(s)Q(s))$.

Youla's Q parameterization—proof

- For any stable Q, $Y(s) = Y_0(s) - M(s)Q(s)$ and $X(s) = X_0(s) - N(s)Q(s)$ are stable, and the proposed controller $K(s) = Y(s)/X(s)$ is stable:

$$M(s)X(s) - N(s)Y(s) = M(s)(X_0(s) - N(s)Q(s)) - N(s)(Y_0(s) - M(s)Q(s))$$
$$= M(s)X_0(s) - M(s)N(s)Q(s) - N(s)Y_0(s) + N(s)M(s)Q(s) = 1.$$

- Conversely, assume $K_1(s) = Y_1(s)/X_1(s)$ is a stabilizing controller, such that $M(s)X_1(s) - N(s)Y_1(s) = 1$. Then

$$\frac{Y_1(s)}{X_1(s)} = \frac{Y_0(s) - M(s)Q(s)}{X_0(s) - N(s)Q(s)}$$

  implies that

$$Y_1(s)X_0(s) - Y_1(s)N(s)Q(s) = X_1(s)Y_0(s) - X_1(s)M(s)Q(s).$$

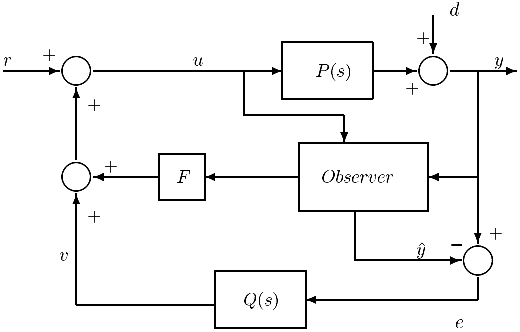
  Rearranging, we get

$$Y_1(s)X_0(s) - X_1(s)Y_0(s) = Y_1(s)N(s)Q(s) - X_1(s)M(s)Q(s) = Q(s).$$

  Since the transfer function on the left is a stable transfer function, this completes the proof.

Youla's Q parameterization — block diagram

- Set $u = F\hat{x} + r + v$, where v is the output of a stable system Q with input $y - \hat{y}$:



- You can show (see, e.g., exercise 29.6 in the textbook) that this block diagram corresponds to the Youla parameterization described previously in algebraic terms.
- This parameterizes all possible stabilizing LTI output feedback controllers, i.e., LTI maps from y to u.