6.241 Dynamic Systems and Control

Lecture 22: Balanced Realization

Readings: DDV, Chapter 26

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Minimal Realizations

- We have seen in the previous lectures how to obtain minimal realizations from non-minimal realizations (i.e., keeping the reachable and observable part from the Kalman decomposition), and also algorithms to construct minimal realizations of a transfer functions.
- Minimal realizations are unique up to similarity transformations.
- However, there are some realizations that are more useful than others, for a number of reasons
  - Kalman decomposition
  - Standard forms
  - Canonical forms
  - ...
- In this lecture we will consider what is known as balanced realization.

The Hankel Operator

- Consider for simplicity a discrete-time system G with state-space realization (A, B, C, D), and transfer function $H(z)$, with impulse response $(H_0, H_1, H_2, \ldots)$.
- How do outputs at time steps $k \geq 0$ depend on inputs at time steps $k < 0$?

$$y_+ = \begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ \vdots \end{bmatrix} = \begin{bmatrix} H_0 & H_1 & H_2 & \cdots \\ H_1 & H_2 & \cdots & \cdots \\ H_2 & \vdots & \ddots & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} u[-1] \\ u[-2] \\ u[-3] \\ \vdots \end{bmatrix} = \mathcal{H} u_-,$$

- the Hankel operator $\mathcal{H}$ transforms past inputs u_- into future outputs y_+.

Structure of the Hankel Operator

- Recall that $H_0 = D$, and $H_k = CA^{k-1}B$. The Hankel operator can be written as

$$\mathcal{H} = \begin{bmatrix} H_0 & H_1 & H_2 & \cdots \\ H_1 & H_2 & \cdots & \cdots \\ H_2 & \vdots & \ddots & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix} \begin{bmatrix} B & AB & A^2B & \cdots \end{bmatrix} = O_\infty R_\infty$$

- Since (A, B, C, D) is a minimal realization, $\mathrm{Rank}(\mathcal{H}) = n$.
- In particular, $\mathcal{H}$ will have exactly n non-zero singular values, which are also called the Hankel singular values of the system G.

Computation of the Hankel singular values

- Recall that, given the properties of singular values,

$$\sigma_i(\mathcal{H}) = \sqrt{\lambda_i(\mathcal{H}\mathcal{H}^T)} = \sqrt{\lambda_i(\mathcal{H}^T\mathcal{H})}.$$

- Notice that
  - $\mathcal{H}\mathcal{H}^T = O_\infty R_\infty R_\infty^T O_\infty^T = O_\infty P O_\infty^T$
  - The (DT) reachability Gramian $P = R_\infty R_\infty^T$ satisfies $APA^T - P = -BB^T$. Similarly, $Q = O_\infty^T O_\infty$, and $A^T Q A - Q = -C^T C$.
- Since $\mathcal{H}\mathcal{H}^T w_i = \sigma_i^2 w_i$ by definition, we also have

$$O_\infty^T \mathcal{H}\mathcal{H}^T w_i = O_\infty^T O_\infty P O_\infty^T w_i = QPO_\infty^T w_i = \sigma_i^2 O_\infty^T w_i.$$

- In other words,

$$\sigma_i(\mathcal{H}) = \sigma_i(PQ), \qquad i = 1, \ldots, n.$$

  the Hankel singular values can be easily computed from the knowledge of the reachability and observability Gramians.

Hankel norm of a system

- Consider bounded-energy "past" input signals $\|u_-\|_2 < \infty$. How much does the energy of the past input get amplified in the energy of the "future" output signal $\|y_+\|_2$?
- This is an induced 2-norm, called the Hankel norm:

$$\|G\|_H := \sup_{\|u_-\|_2 \neq 0} \frac{\|y_+\|_2}{\|u_-\|_2}.$$

- This can be computed easily as $\|G\|_H = \sigma_{\max}(\mathcal{H}) = \sigma_{\max}(PQ)$.
- Note that, for any system G, $\|G\|_H \leq \|G\|_\infty$.
- The state $x[0]$, depending on the realization, separates past and future:
    - The energy necessary to drive the system to $x[0]$ (i.e., $\|u_-\|_2$) is determined by (the inverse of) the reachability Gramian P.
    - The energy in the output from $x[0]$ (i.e., $\|y_+\|_2$) is determined by the observability Gramian Q.
      (Note that $\|y_+\|_2^2 = x[0]^T C^T C x[0] + x[0]^T A^T C^T C A x[0] + \ldots = x[0]^T Q x[0]$, similarly for the control effort.)

Balanced Realization

- It is of interest to "balance" the energy allocation between past control effort and future output energy, i.e., to equalize P and Q.
- A balanced realization is such that $P = Q = \mathrm{diag}(\sigma_1, \sigma_2, \ldots)$.
- Can we find a similarity transformation T such that the realization is balanced?
  - Recall $(A, B, C, D) \to (T^{-1}AT, T^{-1}B, CT, D)$.
  - Gramians are transformed as

$$APA^T - P = -BB^T \quad \to \quad T^{-1}AT\hat{P}T^TA^TT^{-T} - \hat{P} = -T^{-1}BBT^{-T},$$

i.e., $P \to T^{-1}PT^{-T} = \hat{P}$. Similarly, $Q \to T^TQT = \hat{Q}$.

Balanced Realization

- We would like $\hat{P}\hat{Q} = \mathrm{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2) = \Sigma^2$. In other words,
$$T^{-1}PT^{-T}T^TQT = T^{-1}PQT = \Sigma^2.$$
- Since Q is positive definite, one can find a matrix R such that $Q = R^TR$. Hence,
$$T^{-1}PR^TRT = (RT)^{-1}RPR^T(RT) = \Sigma^2$$
- RPR^T is symmetric and positive definite, and can be diagonalized by an orthogonal matrix U, such that
$$RPR^T = U\Sigma^2U^T.$$
- Choose $T = R^{-1}U\Sigma^{1/2}$; then,
$$\hat{P} = \Sigma^{-1/2}U^TRPR^TU\Sigma^{-1/2} = \Sigma,$$
and similarly for Σ.

Model Reduction

- Assume that we have a stable system G, with a minimal realization of order $n >> 1$.
- It is desired to find a reduced-order model (of order $k < n$) in such a way that some “error” is reduced.
- A possible criterion is to find the reduced-order model that minimizes the Hankel norm of the error, i.e., such that $\|G - G^k\|_H$ is minimized.
- Clearly $\|G - G^k\|_H \geq \sigma_{k+1}(\mathcal{H})$.
- It is possible to compute a model that achieves exactly this bound (Glover ’84), but the procedure will not be covered in this course (see, e.g., 6.242).

Model reduction through balanced truncation

- A commonly used procedure for model reduction is based on the balanced realization.
- Idea: remove from the system matrices (in the balanced realization) the blocks corresponding to the smaller Hankel singular values.

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \quad \to \quad G : \begin{bmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ C_1 & C_2 & D \end{bmatrix} \quad \to \quad G^k : \begin{bmatrix} A_{11} & B_1 \\ C_1 & D \end{bmatrix}$$

- If Σ_1 and Σ_2 do not contain any common elements, then the two resulting systems (in particular, the reduced-order model) will be stable.
- We have the following bounds:

$$\sigma_{k+1}(\mathcal{H}) \leq \|G - G^k\|_H \leq \|G - G^k\|_\infty \leq 2 \sum_{l>k} \sigma_l(\mathcal{H}).$$

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