6.241 Dynamic Systems and Control

Lecture 21: Minimal Realizations

Readings: DDV, Chapters 25

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Reachable/(un)observable subspaces

Recall:

- The set of reachable states is a subspace of the state space $\mathbb{R}^n$, given by

$$\mathrm{Ra}(R_n) := \mathrm{Ra}\left(\left[A^{n-1}B|\ldots|AB|B\right]\right).$$

- The set of unobservable states is a subspace of the state space $\mathbb{R}^n$, given by

$$\mathrm{Null}(O_n) := \mathrm{Null}\left(\begin{bmatrix} C \\ CA \\ \ldots \\ CA^{n-1} \end{bmatrix}\right).$$

- Both the reachable space and the unobservable space are A invariant, i.e., if x is reachable (resp., unobservable) so is Ax.

Kalman Decomposition

- Construct an invertible matrix in the following way:

$$T = \begin{bmatrix} T_r & T_{\bar{r}} \end{bmatrix} = \begin{bmatrix} T_{r\bar{o}} & T_{ro} & T_{\bar{r}\bar{o}} & T_{\bar{r}o} \end{bmatrix},$$

  where

  - the columns of $T_r = [T_{r\bar{o}} \, T_{ro}]$ form a basis for the reachable space. In particular, the columns of $T_{r\bar{o}}$ are also in the unobservable space.
  - the columns of $T_{\bar{r}}[T_{\bar{r}\bar{o}} \, T_{\bar{r}o}]$ complement the reachable space. In particular, the columns of $T_{\bar{r}\bar{o}}$ are also in the unobservable space.
  - Note that any of the matrices appearing in the definition of T may in fact have 0 columns, i.e., not be present in particular instances (e.g., for reachable and observable systems, one would only have T_{ro})

- Use the matrix T for a similarity transformation:

$$(A, B, C, D) \to (T^{-1}AT, T^{-1}B, CT, D) = (\hat{A}, \hat{B}, \hat{C}, \hat{D});$$

  this is called the Kalman decomposition.

Kalman Decomposition — structure of the system matrix

- Based on the definition of T, one can write

$$A[T_r \;\; T_{\bar{r}}] = [T_r \;\; T_{\bar{r}}] \begin{bmatrix} A_{rr} & A_{\bar{r}r} \\ A_{r\bar{r}} & A_{\bar{r}\bar{r}} \end{bmatrix}$$

i.e.,

$$A \begin{bmatrix} T_{r\bar{o}} & T_{ro} & T_{\bar{r}\bar{o}} & T_{\bar{r}o} \end{bmatrix} = \begin{bmatrix} T_{r\bar{o}} & T_{ro} & T_{\bar{r}\bar{o}} & T_{\bar{r}o} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}$$

- Since the range of T_r is A-invariant, then $A_{r\bar{r}}$ must be zero, i.e., A_{31}, A_{32}, A_{41}, $A_{42} = 0$.
- Since the range of $[T_{r\bar{o}} \, T_{\bar{r}\bar{o}}]$ is A-invariant, then A_{21}, A_{23}, A_{41}, A_{43} must also be zero.

Kalman Decomposition — structure of the B, C matrices

- Noting that $\mathrm{Ra}(B) \in \mathrm{Ra}(R_n)$, and

$$B = T\hat{B} = [T_r \, T_{\bar{r}}] \begin{bmatrix} B_r \\ B_{\bar{r}} \end{bmatrix},$$

  one can conclude that $B_{\bar{r}} = 0$, i.e., $\hat{B} = \begin{bmatrix} B_r \\ 0 \end{bmatrix}$.

- Similarly, since $\mathrm{Null}(O_n) \subseteq \mathrm{Null}(C)$, and

$$CT = C \begin{bmatrix} T_{r\bar{o}} & T_{ro} & T_{\bar{r}\bar{o}} & T_{\bar{r}o} \end{bmatrix} = \hat{C},$$

  one can conclude that $C_{r\bar{o}}$, $C_{\bar{r}\bar{o}}$ must be zero, i.e.,

$$\hat{C} = \begin{bmatrix} 0 & C_{ro} & 0 & C_{\bar{r}o} \end{bmatrix}.$$

Kalman Decomposition

- Summarizing, we get

$$\hat{A} = \begin{bmatrix} A_{r\bar{o}} & A_{12} & A_{13} & A_{14} \\ 0 & A_{ro} & 0 & A_{24} \\ 0 & 0 & A_{\bar{r}\bar{o}} & A_{34} \\ 0 & 0 & 0 & A_{\bar{r}o} \end{bmatrix}, \qquad \hat{B} = \begin{bmatrix} B_{r\bar{o}} \\ B_{ro} \\ 0 \\ 0 \end{bmatrix}$$

$$\hat{C} = \begin{bmatrix} 0 & C_{ro} & 0 & C_{\bar{r}o} \end{bmatrix}, \qquad \hat{D} = D.$$

- From this decomposition, one can get the reachable subsystem:

$$\left(\begin{bmatrix} A_{r\bar{o}} & A_{12} \\ 0 & A_{ro} \end{bmatrix}, \begin{bmatrix} B_{r\bar{o}} \\ B_{ro} \end{bmatrix}, \begin{bmatrix} 0 & C_{ro} \end{bmatrix}, D \right),$$

and the observable subsystem

$$\left(\begin{bmatrix} A_{ro} & A_{24} \\ 0 & A_{\bar{r}o} \end{bmatrix}, \begin{bmatrix} B_{ro} \\ 0 \end{bmatrix}, \begin{bmatrix} C_{ro} & C_{\bar{r}o} \end{bmatrix}, D \right),$$

with their unobservable/uncontrollable parts clearly displayed.

Remarks on the Kalman decomposition

—figure showing input-output connections—

- The Kalman decomposition is unique up to similarity transformation with the same block structure.
- Eigenvalues of the various subsystems are uniquely defined.

Realizations

- Recall that given a discrete-time state-space model (A, B, C, D), one can obtain an equivalent I/O model with transfer function $H(z) = C(zI - A)^{-1}B + D$.
- How can we do the converse? i.e., given a transfer function, how can we get an equivalent state-space model?
- Note that

$$H(z) = C(zI - A)^{-1}B + D = C_{ro}(zI - A_{ro})^{-1}B_{ro} + D,$$

  i.e., the transfer function of a system is entirely defined by its reachable and observable part.
- The function $H(z)$ can also be written as

$$H(z) = H_0 + z^{-1}H_1 + z^{-2}H_2 + \dots,$$

  where the coefficients H_i (also called the Markov parameters) describe the response at time step i to an impulse at time 0 (and zero initial conditions). These coefficients can be computed as

$$H_0 = D, \text{ and } H_k = CA^{k-1}B, \text{ for } k \geq 1.$$

Minimal Realizations

- In particular, one is interested in getting the smallest possible realization of a transfer function model.
- Theorem: A realization is minimal if and only if it is reachable and observable.
- **Proof**:
  - For the necessity part, it is clear that if a realization of a transfer function is not reachable or not observable, one could extract its reachable and observable part through the Kalman decomposition, which is smaller.
  - For sufficiency, assume (A, B, C, D) is reachable and observable of order n, but is not minimal, i.e., there is another (reachable and observable) realization (A^*, B^*, C^*, D^*) of smaller order n^*. Then,

$$O_n R_n = \begin{bmatrix} H_1 & H_2 & \dots & H_n \\ H_2 & H_3 & \dots & \\ \dots & & & \\ H_n & \dots & \dots & H_{2n-1} \end{bmatrix} = O_n^* R_n^*$$

    but the rank of $O_n R_n$ is n, while the rank of $O_n^* R_n^*$ is $n^* < n$, which is a contradiction.

Minimal Realizations of SISO systems

- A way to compute a minimal realization of a SISO system is by using canonical forms, e.g., controller canonical form.
- In this case, given a (proper) rational transfer function in the form

$$G(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \ldots + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} \ldots + a_0} + G(\infty),$$

we get

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & & & & \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ \ldots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} b_0 & b_1 & \ldots & b_{n-1} \end{bmatrix}, \qquad D = G(\infty).$$

Minimal Realizations of MIMO systems

- Could do a SISO minimal realization for each entry in the matrix transfer function.

- However, this realization may not be minimal

Gilbert's realization

- Consider a matrix transfer function $H(z)$, with m inputs and p outputs.
- Let $d(s)$ be the least common denominator, and assume that $d(z)$ has no repeated roots.
- Compute the partial fraction expansion of H, in the form $H(z) = H(\infty) + \sum_{i=1}^{q} \frac{1}{z-\rho_i} R_i$; let r_i be the rank of each residue matrix R_i.
- Write each residue matrix as $R_i = C_i^{p \times r_i} B_i^{r_i \times m}$
- The desired realization is:

$$A = \begin{bmatrix} \rho_1 & & & & & & \\ & \ddots & & & & & \\ & & \rho_1 & & & & \\ & & & \ddots & & & \\ & & & & \rho_i & & \\ & & & & & \ddots & \\ & & & & & & \rho_i \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ \vdots \\ B_i \end{bmatrix}$$

$$C = \begin{bmatrix} C_1 & \cdots & C_i \end{bmatrix}, \qquad D = H(\infty).$$

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