6.241 Dynamic Systems and Control

Lecture 20: Reachability and Observability

Readings: DDV, Chapters 23, 24

Emilio Frazzoli

Aeronautics and Astronautics
Massachusetts Institute of Technology

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Reachability in continuous time

- Given a system described by the (n-dimensional) state-space model

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = 0,$$

a point x_d is said to be reachable in time L if there exists an input $u : t \in [0, L] \mapsto u(t)$ such that $x(L) = x_d$.

- Given an input signal over $[0, L]$, one can compute

$$x(L) = \int_0^L e^{A(L-t)} Bu(t)\, dt = \int_0^L F^T(t) u(t)\, dt =: \prec F, u \succ_L,$$

where $F^T(t) := e^{A(L-t)} B$.

- The set $\mathcal{R}$ of all reachable points is a linear (sub)space: if x_a and x_b are reachable, so is $\alpha x_a + \beta x_b$.
- If the reachable set is the entire state space, i.e., if $\mathcal{R} = \mathbb{R}^n$, then the system is called reachable.

Reachability Gramian

Theorem

Let $\mathcal{P}_L := \prec F, F \succ = \int_0^L F^T(t) F(t)\, dt$. *Then,*

$$\mathcal{R} = \mathrm{Ra}(\mathcal{P}_L).$$

- Prove that $\mathcal{R} \subseteq \mathrm{Ra}(\mathcal{P}_L)$, i.e., $\mathcal{R}^\perp \supseteq \mathrm{Ra}^\perp(\mathcal{P}_L)$.
- $q^T \mathcal{P}_L = 0 \Rightarrow q^T \mathcal{P}_L q = 0 \Leftrightarrow \prec Fq, Fq \succ = 0 \Leftrightarrow q^T F^T(t) = 0 \Rightarrow q^T x(L) = 0$ (i.e., if $q \in \mathrm{Ra}^\perp(\mathcal{P}_L)$, then $q \in \mathcal{R}^\perp$.)
- Now prove that $\mathcal{R} \supseteq \mathrm{Ra}(\mathcal{P}_L)$: let α be such that $x_d = \mathcal{P}_L \alpha$, and pick $u(t) = F(t)\alpha$. Then

$$x(L) = \int_0^L F^T(t) F(t) \alpha\, dt = \mathcal{P}_L \alpha = x_d.$$

Reachability Matrix

Theorem

$$\mathrm{Ra}(\mathcal{P}_L) = \mathrm{Ra}\left([A^{n-1}B|\dots|AB|B]\right) = \mathrm{Ra}(R_m).$$

- $q^T \mathcal{P}_L = 0 \Rightarrow q^T e^{A(L-t)} B = 0 \Rightarrow q^T R_n = 0$
- $q^T R_n \Leftrightarrow q^T A^l B = 0, \forall l \in \mathbb{N} \Rightarrow q^T(t) e^{A(L-t)} B = 0 \forall t \in \mathbb{R} \Rightarrow q^T \mathcal{P}_L = 0.$
- The system is reachable iff the rank of R_n is equal to n.
- Notice that this condition does not depend on L!
- Reachability vs. Controllability: a state x_d is controllable if one can find a control input u such that

$$e^{AL} x_d + \prec F, u \succ_L = 0.$$

This is equivalent to $x_d = e^{-AL} \prec F, u \succ_L$, i.e., controllability and reachability coincide for CT systems. (They do not coincide for DT systems, e.g., is the matrix A is not invertible.)

Computation of the reachability Gramian

- Recall the definition of the reachability Gramian at time L:

$$\mathcal{P}_L := \int_0^L e^{A(L-t)} B B^T e^{A(L-t)^T} \, dt = \int_0^L e^{A\tau} B B^T e^{A^T \tau} \, d\tau$$

- Recall that the range does not depend on L. In particular, assuming the system is stable (i.e., all eigenvalues of A are in the open left half plane), one can consider $L \to +\infty$, and define

$$\mathcal{P} := \lim_{L \to +\infty} \mathcal{P}_L = \int_0^{+\infty} e^{A\tau} B B^T e^{A^T \tau} \, d\tau$$

Computation of the reachability Gramian

Theorem

The reachability Gramian satisfies the Lyapunov equation

$$A\mathcal{P} + \mathcal{P}A^T = -BB^T.$$

- $\int_0^\infty \frac{d}{dt}\left(e^{At}BB^T e^{A^T t}\right)\, dt = -BB^T$
- $\int_0^\infty \frac{d}{dt}\left(e^{At}BB^T e^{A^T t}\right)\, dt = A\mathcal{P} + \mathcal{P}A^T.$

Canonical forms

- Consider the similarity transformation $AR = R\bar{A}$, $b = R\bar{b}$.
- Using Cayley-Hamilton, we get

$$\bar{A} = \begin{bmatrix} -a_{n-1} & 1 & 0 & \dots \\ -a_{n-2} & 0 & 1 & \dots \\ \dots \\ -a_0 & 0 & 0 & \dots \end{bmatrix}, \qquad \bar{b} = \begin{bmatrix} 0 \\ 0 \\ ldots \\ 1 \end{bmatrix},$$

called the controllability form.

- A similar transformation can be used to take the system to the controller canonical form...

Observability

- Given a system described by the (n-dimensional) state-space model

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t),$$

a state $x_q \neq 0$ is said unobservable over $[0, L]$, if for every input u, the output y_q obtained with initial condition $x(0) = x_q$ is the same as the output y_0 obtained with initial condition $x(0) = 0$.

- A dynamic system is said unobservable if it contains at least an unobservable state, observable otherwise.

- Note that observability can be established assuming zero input.

Continuous-time observability

- For continuous-time systems, the following are equivalent:
  - x_q is unobservable in time L
  - x_q is unobservable in any time.
  - $O_n x_q = \begin{bmatrix} C \\ CA \\ \dots \\ CA^{n-1} \end{bmatrix} x_q = 0.$
- 1) $\Rightarrow$ 2): if x_q unobservable in time L, then for $u = 0$ $y = 0$, i.e., $Ce^{At}x_q = 0$ for all $t \in [0, L]$. hence, $Ce^{A\cdot 0}x_q = 0$, $d/dtCe^{At}x_q|_{t=0} = CAx_q = 0$... which implies that $Ce^{At}x_q = 0$ for all $t > L$ as well.
- 2) $\Rightarrow$ 1): immediate
- 2) $\Leftrightarrow$ 3): Cayley-Hamilton implies that $\mathrm{Null} O_n = \mathrm{Null} \begin{bmatrix} C \\ CA \\ \dots \\ CA^{l-1} \end{bmatrix}$ for all $l > n$.

Observability Gramian

- Define the observability Gramian at time L:

$$\mathcal{Q}_L := \int_0^L e^{A^T(L-t)} C^T C e^{A(L-t)}\, dt = \int_0^L e^{A^T\tau} C^T C e^{A\tau}\, d\tau$$

- Recall that observability does not depend on L. In particular, assuming the system is stable (i.e., all eigenvalues of A are in the open left half plane), one can consider $L \to +\infty$, and define

$$\mathcal{Q} := \lim_{L\to+\infty} \mathcal{Q}_L = \int_0^{+\infty} e^{A^T\tau} C^T C e^{A\tau}\, d\tau$$

Theorem

The observability Gramian satisfies the Lyapunov equation

$$A^T\mathcal{Q} + \mathcal{Q}A = -C^T C.$$

other results

- Essentially, observability results are similar to their reachability counterparts, when considering (A^T, C^T) as opposed to (A, B). In particular,
- (A, C) is unobservable if $Cv_i = 0$ for some (right) eigenvector v_i of A. $y(t) = C \sum i = 1^n e^{\lambda_i t} v_i w_i^T x(0)$
- (A, C) is unobservable if $\begin{bmatrix} sI - A \\ C \end{bmatrix}$ drop ranks for some $s = \lambda$, this λ is an unobservabe eigenvalue for the system.
- The dual of controllability to the origin is "constructability": same considerations as in the reachability/controllability case hold.

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